Polynomial Relations among Principal Minors of a Matrix

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Problem Description

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  $A_* \in \mathbb{C}^{2^n}$ – vector whose entries are the principal minors of A.

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$$I = 124 \qquad \begin{pmatrix} a_{11} & a_{12} & \cdot & a_{14} \\ a_{21} & a_{22} & \cdot & a_{24} \\ \cdot & \cdot & \cdot & \cdot \\ a_{41} & a_{42} & \cdot & a_{44} \end{pmatrix} \qquad A_{124} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

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Note that $(A, \mathrm{Id}_n)_I = A_I$. Define vector $(A,B)_* \in \mathbb{C}^{2^n}$.

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- Principal Minors of Symmetric Matrix
  - Holtz & Sturmfels [3], 2007: studied relations among principal minors of a *symmetric* 4×4 matrix.
  - Found links to hyperdeterminant.
  - Oeding [6], 2008: found set-theoretic defining equations for all n.

Cycles and Cycle-sums

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Let $A = (a_{ij})$ be a complex $n \times n$ matrix.

- Given a cyclic permutation $\pi = (i_1 \dots i_k) \in \mathfrak{S}_n$, define *cycle*

$$c_\pi = a_{i_1\pi(i_1)} a_{i_2\pi(i_2)} \cdots a_{i_k\pi(i_k)}$$

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- Given a subset $I \subseteq [n]$, $|I| \geq 2$, define *cycle-sum*

$$C_I = \sum_{\pi \in \mathfrak{C}_I} c_\pi$$

over the set $\mathfrak{C}_I$ of cyclic permutations with support I. Also define $C_\emptyset = 1$, $C_{\{i\}} = a_{ii}$, giving 2^n cycle-sums.

Principal Minors and Cycle-sums

Prop 1. The principal minors and cycle-sums satisfy

$$A_I = \sum_{I = I_1 \sqcup \ldots \sqcup I_k} (-1)^{k+d} C_{I_1} \cdots C_{I_k}$$

$$C_I = \sum_{I = I_1 \sqcup \ldots \sqcup I_k} (-1)^{k+d} (k-1)! A_{I_1} \cdots A_{I_k}$$

where $I \subset [n]$, $|I| = d$ and $I_1 \sqcup \ldots \sqcup I_k$ are partitions of I.

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Cor 2. Let $\psi : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$, $A_* \mapsto C_*$. Then $u \in \mathbb{C}^{2^n}$ is realizable as principal minors iff $\psi(u)$ is realizable as cycle-sums.

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Cor 3. The subrings $\mathbb{C}[A_*]$ and $\mathbb{C}[C_*]$ of $\mathbb{C}[(a_{ij})]$ are equal.

Closure Theorem

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Proof Idea.

1. $\mathbb{C}[c_*]$ is integral over $\mathbb{C}[C_*] = \mathbb{C}[A_*]$.
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Cor 5. A vector $u \in \mathbb{C}^{2^n}$ is realizable as principal minors iff it satisfies the relations in $\mathcal{I}_n$.

i.e. PMAP is solved if we find finite generating sets for $\mathcal{I}_n$.

Finding Affine Relations in $\mathcal{I}_4$

Small Cases

- For $n \leq 3$, all vectors of length 2^n are realizable as principal minors.

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- For $n = 4$, this vector is not realizable:

$$A_{123} = A_{124} = A_{134} = A_{234} = 1$$
$$A_1 = A_2 = \ldots = A_{1234} = 0$$

By the Closure Theorem,

$$\mathcal{I}_4 \neq \{0\}$$

Some relations were found by Nanson [4,5] in 1897.

Nanson Relations

We express the Nanson relations in terms of cycle-sums. They are the 4×4 minors of this 5×4 matrix.

$$\begin{pmatrix} C_{123}C_{14} & C_{124}C_{13} & C_{134}C_{12} & 2C_{234}C_{12}C_{13}C_{14} + C_{134}C_{124}C_{123} \\ C_{124}C_{23} & C_{123}C_{24} & C_{234}C_{21} & 2C_{134}C_{21}C_{23}C_{24} + C_{234}C_{124}C_{123} \\ C_{134}C_{32} & C_{234}C_{31} & C_{123}C_{34} & 2C_{124}C_{31}C_{32}C_{34} + C_{234}C_{134}C_{123} \\ C_{234}C_{41} & C_{134}C_{42} & C_{124}C_{43} & 2C_{123}C_{41}C_{42}C_{43} + C_{234}C_{134}C_{124} \\ 1 & 1 & 1 & C_{1234} \end{pmatrix}$$

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In cycle-sums:

4 poly (deg 8 and 32 terms), 1 poly (deg 10 and 19 terms)

In principal minors:

4 poly (deg 12 and 5234 terms), 1 poly (deg 16 and 19012 terms)

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Let I be the ideal generated by these maximal minors.

The vector $C_{123} = C_{124} = C_{134} = C_{234} = 1$,

$C_1 = C_2 = \ldots = C_{1234} = 0$ is not realizable,

but satisfies the Nanson relations. Thus, $I \neq \mathcal{I}_4$.

Affine Relations $\mathcal{I}_4$

Define $g = \begin{vmatrix} C_{123}C_{14} & C_{124}C_{13} & C_{134}C_{12} \\ C_{124}C_{23} & C_{123}C_{24} & C_{234}C_{21} \\ C_{134}C_{32} & C_{234}C_{31} & C_{123}C_{34} \end{vmatrix}$

Thm 6. The ideal $\mathcal{I}_4$ is the ideal quotient $(I : g)$.
It is minimally generated by 65 polynomials of deg 12.

Finding Projective Relations in $\mathcal{J}_4$ using Lie Algebras

Lie Group Action

- Isomorphism between $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and vector space generated by principal minors. e.g.

$$A_{123} \quad \leftrightarrow \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$aA_{23} + bA_{123} \quad \leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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$$

- Action of $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ induces action on $\mathbb{C}[A_*]$.

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- Action of $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ induces action on $\mathbb{C}[A_*]$.
- $\mathcal{J}_4$ is invariant under this Lie group action.

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- Hence, $\operatorname{Im}\phi$ is cut out scheme-theoretically by ideal $\mathcal{K}$ generated by orbit of homogenizations of 65 degree-12 affine generators.
- $\mathcal{K}$ is generated by degree-12 component $\mathcal{K}_{12}$. We want to find the module structure of $\mathcal{K}_{12}$, e.g. what are the irreducible components?

Lie Algebra Action

- Lie group action of $\mathrm{GL}_2(\mathbb{C})^4$ induces Lie algebra action of $\mathfrak{gl}_2(\mathbb{C})^4$ on $\mathbb{C}[A_*]$ by differential operators. e.g.

$$(\mathbf{0}, \mathbf{0}, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \mathbf{0})$$ acts as

$$\sum_{3 \notin I} \left(wA_I \frac{\partial}{\partial A_I} + xA_{I \cup \{3\}} \frac{\partial}{\partial A_I} + yA_I \frac{\partial}{\partial A_{I \cup \{3\}}} + zA_{I \cup \{3\}} \frac{\partial}{\partial A_{I \cup \{3\}}} \right)$$

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- Applying the *raising operators* on the 65 homogenized affine generators, we find the *highest weight vectors* of the irreducible components of $\mathcal{K}_{12}$.

Projective Relations $\mathcal{J}_4$

Thm 7. $\operatorname{Im}\phi$ is cut out scheme-theoretically by 718 linearly independent homogeneous polynomials of degree 12.

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This space of polynomials is the direct sum of three irreducible $\mathfrak{S}_4 \times \mathrm{GL}_2(\mathbb{C})^4$-modules

$$M_D \oplus M_E \oplus M_F$$

which are orbits of polynomials D, E, F respectively.

D, E, F have $32, 42, 91$ terms (in cycle-sums) respectively.
D is the Nanson relation.

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Conj 8. $\mathcal{J}_4$ is generated by these 718 polynomials.

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After-note:
Eric Rains points out to us that in his paper [1] with A. Borodin on determinantal point processes, they found 718 degree 12 generators for $\mathcal{J}_4$.
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Conj 9. For $n > 4$, $\mathcal{J}_n$ is generated by the $\mathfrak{S}_n \times \mathrm{GL}_2(\mathbb{C})^n$-orbit of deg 12 polynomials D, E, F.

Hyperdeterminantal Relations

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Define
$$F = A_\emptyset + A_1 x + A_2 y + A_3 z + A_4 w$$
$$+A_{12}xy + A_{13}xz + A_{14}xw + A_{23}yz + A_{24}yw + A_{34}zw$$
$$+A_{123}xyz + A_{124}xyw + A_{134}xzw + A_{234}yzw + A_{1234}xyzw.$$

Def 10. The hyperdeterminant D_{2222} is the unique irreducible polynomial (up to sign) of content one in the unknowns A_* which vanishes whenever

$$F = \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = \partial F/\partial w = 0$$

has a solution (x_0, y_0, z_0, w_0) in $\mathbb{C}^4$.

Hyperdeterminantal Relations

The irreducible components of the singular locus ∇_{sing} of the hypersurface $D_{2222} = 0$ were classified by Weyman and Zelevinsky [7] in 1996.

$$\nabla_{\text{sing}} \;=\; \nabla_{\text{node}}(\emptyset) \;\cup \bigcup_{1 \le i < j \le 4} \nabla_{\text{node}}(\{i,j\}) \;\cup\; \nabla_{\text{cusp}}$$

Thm 11. $\operatorname{Im}\phi$ is the irreducible component $\nabla_{\text{node}}(\emptyset)$.

Summary

- Usefulness of cycles and cycle-sums for determinantal problems.
  - Closure of $\mathrm{Im}\,\phi_a$ and $\mathrm{Im}\,\phi$.
  - Nanson relations in $\mathcal{I}_4$.
- Exploiting symmetries.
  - Using Lie group, Lie algebra action to find $\mathcal{J}_4$.
- Relation to hyperdeterminants.

`http://math.berkeley.edu/~shaowei/minors.html`

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