Tropical Implicitization

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"Elimination Theory for Tropical Varieties" Sturmfels, Tevelev

Problem

$f_1, \ldots, f_s \in \mathbb{C}[t_1^{\pm}, \ldots, t_r^{\pm}]$ Laurent polynomials

$f = (f_1, \ldots, f_s) : \mathbb{T}^r \to \mathbb{T}^s$ rational map

$Y = \overline{\mathrm{Im} f}$

Compute tropical variety $\mathcal{T}(Y)$

First, we recall a theorem from Bernd's class on tropical elimination.

Thm. $X \subset \mathbb{T}^n$ closed subvariety

$\alpha : \mathbb{T}^n \to \mathbb{T}^d$ homomorphism of tori
generically finite from X onto $\alpha(X)$

$A : \mathbb{Z}^n \to \mathbb{Z}^d$ corr. linear map of lattices

Then $\mathcal{T}(\alpha(X)) = A(\mathcal{T}(X))$

Approach

Consider graph $X \subset \mathbb{T}^{r+s}$ of f with coords $(t_1, \ldots, t_r, y_1, \ldots, y_s)$

X defined by $f_1(t) - y_1 = \ldots = f_s(t) - y_s = 0$

$\alpha : \mathbb{T}^{r+s} \to \mathbb{T}^s, (t, y) \mapsto y$ projection, A projection

Then $\alpha(X) = Y$ so $\mathcal{T}(Y) = A(\mathcal{T}(X))$

Qn: How to compute $\mathcal{T}(X)$?

Thm (Geometric Tropicalization, Hacking-Keel-Tevelev)

This is the del Pezzo paper. Keep theorem on board.

$X \subset \mathbb{T}^n$ smooth variety with coords $(t_1, \ldots, t_n)$

$\overline{X} \supset X$ compactification, bdy divisor $D = \overline{X} \setminus X$ s.n.c.

$D_1, \ldots, D_m$ irreducible components of D

$\Delta_{X,D}$ simplicial complex on $\{1, 2, \ldots, m\}$
with $\{i_1, \ldots, i_l\} \in \Delta_{X,D}$ iff $D_{i_1} \cap \cdots \cap D_{i_l} \neq \emptyset$

$[D_i] = (\mathrm{val}_{D_i}(t_1), \ldots, \mathrm{val}_{D_i}(t_n)) \in \mathbb{R}^n$

For face $\sigma \in \Delta_{X,D}$, $[\sigma]$ = cone spanned by $\{[D_i] : i \in \sigma\}$

Then $\mathcal{T}(X) = \bigcup_{\sigma \in \Delta_{X,D}} [\sigma]$

Example

Let $Y = \{(u, v) \in \mathbb{T}^2 : v^2 - u^3 - u^2\}$. Find $\mathcal{T}(Y)$.

We know the answer to this.

[Draw Newton Polytope and Inner Normal Fan]

Now, lets do it by tropical implicitization.

Let $f : \mathbb{T}^1 \setminus \{-1, 1\} \to \mathbb{T}^2, t \mapsto (u, v) = (t^2 - 1, t^3 - t)$ so $Y = \overline{\mathrm{Im} f}$

Let $X \subset \mathbb{T}^3$ graph of f

[Draw graph X]

Compactify $\overline{X} \supset X$ via $\mathbb{P}^3 \supset \mathbb{T}^3$

Four bdy divisors (in proj coords)

$$\begin{array}{ccccccccc} D_1 & = & (& 0, & -1, & 0, & 1 &) & = & (t) \\ D_2 & = & (& 1, & 0, & 0, & 1 &) & = & (t-1) \\ D_3 & = & (& -1, & 0, & 0, & 1 &) & = & (t+1) \\ D_4 & = & (& 0, & 0, & 1, & 0 &) & = & (\frac{1}{t}) \end{array}$$

Coordinate divisors

$$\begin{array}{rcrrrr} (t) & = & D_1 & & & -D_4 \\ (u) = (t^2 - 1) & = & & D_2 & +D_3 & -2D_4 \\ (v) = (t^3 - t) & = & D_1 & +D_2 & +D_3 & -3D_4 \end{array}$$

Divisorial rays

$$\begin{array}{cccrrr} [D_1] & = & (& 1, & 0, & 1) \\ [D_2] & = & (& 0, & 1, & 1) \\ [D_3] & = & (& 0, & 1, & 1) \\ [D_4] & = & (& -1, & -2, & -3) \end{array}$$

$\Delta_{X,D}$ consists of 4 points

$\mathcal{T}(X)$ generated by $(0, 1), (1, 1), (-2, -3)$

Algorithm

(need not be generic complete intersection)

Input: $f = (f_1, \ldots, f_s) : \mathbb{T}^r \dashrightarrow \mathbb{T}^s$ rational map

Output: the set $\mathcal{T}(Y)$, $Y = \overline{\mathrm{Im} f}$

(1) $X := \mathbb{T}^r \setminus \cup_{i=1}^s E_i, \quad E_i := \{f_i = 0\}$

(2) Find compactification $\overline{X} \supset X$ smooth with s.n.c.

(3) For each irred bdy divisor D, compute

$$[D] = (\mathrm{val}_D(f_1), \ldots, \mathrm{val}_D(f_s)) \in \mathbb{R}^s$$

(4) Compute $\Delta_{X,D}$

(5) $\mathcal{T}(Y) = \bigcup_{\sigma \in \Delta_{X,D}} [\sigma]$

Example

$f = (f_0, f_1, \ldots, f_6) : \mathbb{T}^6 \to \mathbb{T}^7$

$f_i = a_0 b_0^i b_1^{6-i} + a_1 c_0^i c_1^{6-i}$ secant of Veronese variety, dim = 4

relations on $\widetilde{Y} = \overline{\mathrm{Im} f}$ gen by 3×3 minors of Hankel matrix

$$\begin{pmatrix} f_0 & f_1 & f_2 & f_3 \\ f_1 & f_2 & f_3 & f_4 \\ f_2 & f_3 & f_4 & f_5 \\ f_3 & f_4 & f_5 & f_6 \end{pmatrix}$$

$\mathcal{T}(\widetilde{Y})$ dim = 4, dim(linearity space)=2

("Computing Tropical Varieties" BJSST)

[Draw graph of $\mathcal{T}(\widetilde{Y})$]

Reparametrization

$f_i = \lambda \omega^i (a + c^i)$

Now, we consider instead

$f_i = a + c^i$, $Y = \overline{\mathrm{Im} f}$

Then, modulo the linearity space, $\mathcal{T}(\widetilde{Y}) \equiv \mathcal{T}(Y)$

We compute $\mathcal{T}(Y)$ using the algorithm.

[Draw E_i on graph]

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(Signs opp of that of Gfan)

RAYS
1 0 0 0 0 0 0  # 0  (0)  E0
0 1 0 0 0 0 0  # 5  (6)  E1
0 0 1 0 0 0 0  # 3  (9)  E2
0 0 0 1 0 0 0  # 7  (11) E3
0 0 0 0 1 0 0  # 4  (10) E4
0 0 0 0 0 1 0  # 14 (7)  E5
0 0 0 0 0 0 1  # 1  (1)  E6

0 1 1 1 1 1 1  # 10 (2)  D1
0 1 2 2 2 2 2  # 6  (4)  D2
0 1 2 3 3 3 3  # 2  (8)  D3
0 1 2 3 4 4 4  # 5  (5)  D4
0 1 2 3 4 5 5  # 9  (3)  D5

1 0 1 0 1 0 1  # 11 (13) FE
0 1 0 1 0 1 0  # 12 (14) FO

LEFTOVERS
1 0 0 1 0 0 1  # 8  (12) Divisor at Infty?
0 1 0 3 0 1 0  # 13 (15) Extra?
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