Tropical Implicitization and the Hankel Matrix

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Abstract

We describe the theory and algorithms behind non-generic tropical implicitization using geometric tropicalization. The tropical variety given in [1, Fig 2] of the ideal generated by the 3×3 minors of a 4×4 Hankel matrix is explained in this framework.

1 Introduction

Let $\mathbb{T}^r = (\mathbb{C}^*)^r$ be the r-dimensional algebraic torus over $\mathbb{C}$. The classical implicitization problem is as follows: given Laurent polynomials $f_1, \ldots, f_s \in \mathbb{C}[t_1^\pm, \ldots, t_r^\pm]$ and the rational map $f = (f_1, \ldots, f_s) : \mathbb{T}^r \to \mathbb{T}^s$, find the Zariski closure Y of the image of f. Often, this refers to computing the ideal I_Y of Y in the Laurent polynomial ring $\mathbb{C}[\mathbb{T}^s] = \mathbb{C}[y_1^\pm, \ldots, y_s^\pm]$. For simplicity, we will assume that the fiber of f over a generic point of Y is finite. In this report, we are interested in computing the tropicalization

$$\mathcal{T}(Y) = \{v \in \mathbb{Q}^s : 1 \notin \mathrm{in}_v(I_Y)\}$$

where $\mathrm{in}_v(I_Y)$ is the ideal of all initial forms $\mathrm{in}_v(f)$ for $f \in I_Y$. This tropical implicitization problem can provide crucial information for the classical case. For instance, when Y is a codimension-one hypersurface, $I_Y = \langle g \rangle$ is principal and $\mathcal{T}(Y)$ is the union of non-maximal cones in the normal fan of the Newton polytope of g, so knowing $\mathcal{T}(Y)$ can help us in finding g. Thus, we will want to compute $\mathcal{T}(Y)$ without explicitly knowing I_Y.

When the coefficients of $f_1, \ldots, f_s$ are generic with respect to their Newton polytopes $P_1, \ldots, P_s$, an algorithm for constructing $\mathcal{T}(Y)$ from $P_1, \ldots, P_s$ was given in [6, Thm 2.1] and proved in [5, Thm 5.1]. The authors also formulated

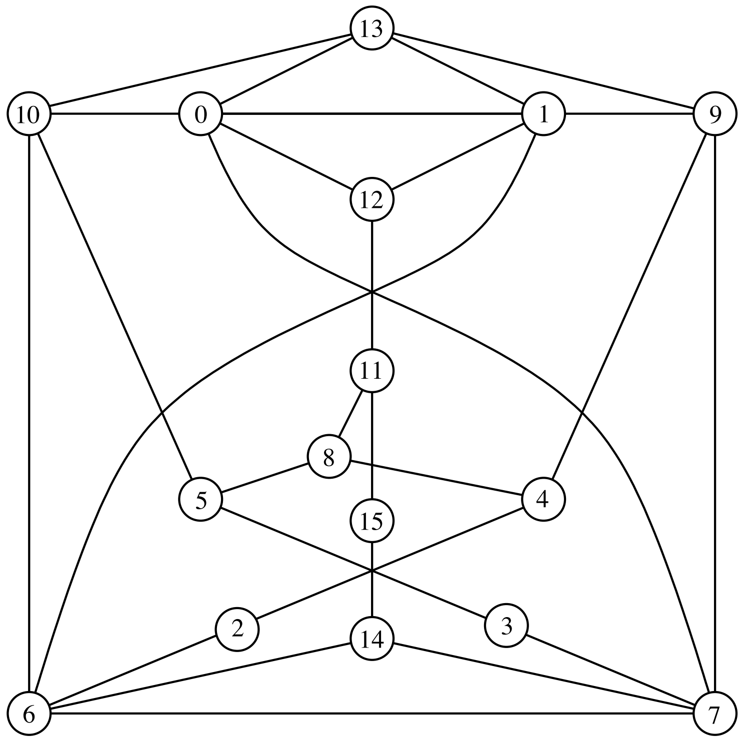


Figure 1: Tropical variety described by minors of a Hankel matrix

a rule for assigning multiplicities to regular points on $\mathcal{T}(Y)$ regardless of the genericity of f. We will not investigate this rule nor explore the generic construction in our report. Rather, we will elaborate on the algorithm proposed in [5, §5] which involves compactifications of classical varieties and divisorial characterizations of tropical varieties. The latter is sometimes referred to as *geometric tropicalization*.

Our goal at the end of the day is to explain Figure 1 from [1, Fig 2] representing the tropical variety $\mathcal{T}(Y) \subset \mathbb{Q}^7$ for the ideal $I_Y \subset \mathbb{C}[y_0^{\pm}, y_1^{\pm}, \ldots, y_6^{\pm}]$ generated by the 3×3 minors of the Hankel matrix

$$\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & y_6 \end{pmatrix}.$$

This tropical variety is 4-dimensional with a 2-dimensional linearity space.

Modulo the linearity space, we intersect $\mathcal{T}(Y)$ with a unit sphere to get the above 1-dimensional simplicial complex where the numbered nodes represent vertices and the lines represent edges. In [1], $\mathcal{T}(Y)$ was computed from the ideal I_Y using the software `Gfan`.

We compute $\mathcal{T}(Y)$ using geometric tropicalization instead. According to [4, Ex 1.2], the variety Y defined by I_Y is also the closure of the image of the rational map $f = (f_0, f_1, \ldots, f_6) : \mathbb{T}^6 \to \mathbb{T}^7$ where

$$f_i = a_0 b_0^i b_1^{6-i} + a_1 c_0^i c_1^{6-i} \in \mathbb{C}[a_0^\pm, a_1^\pm, b_0^\pm, b_1^\pm, c_0^\pm, c_1^\pm].$$

We will see that after accounting for the linearity space, it suffices to study the parametrization $f_i = a + c^i$ for $i = 0, 1, \ldots, 6$. By geometric tropicalization, the nodes $0, 6, 9, 11, 10, 7, 1$ in the diagram come from boundary divisors defined by $f_0, f_1, \ldots, f_6$ respectively. As for the nodes $2, 4, 8, 5, 3$, they are derived from exceptional divisors in the blow up of the origin $(a, c) = (0, 0)$ while the nodes $13, 14, 12$ come from blowing up $(-1, -1)$, $(1, -1)$ and $(-1, \omega)$ respectively, where ω is a primitive third root of unity. Node 15 is extraneous.

This report is organized as follows. In Section 2, we describe the theory and algorithms. In Sections 3 and 4, we demonstrate the algorithms with the tropicalization of the nodal cubic and the explanation of Figure 1.

2 Geometric Tropicalization

We begin with the following useful theorem in tropical elimination theory which may be found in [7, Prop 3.1], [5, Thm 1.1] and [2, Thm 3.1].

Theorem 1. *Let $X \subset \mathbb{T}^n$ be a closed subvariety and $\alpha : \mathbb{T}^n \to \mathbb{T}^d$ a homomorphism of tori generically finite from X onto $\alpha(X)$. Let $A : \mathbb{Z}^n \to \mathbb{Z}^d$ be the linear map of lattices corresponding to α. Then, $\mathcal{T}(\alpha(X)) = A(\mathcal{T}(X))$.*

This theorem can be applied directly to our tropical implicitization problem if the rational map f is a homomorphism of tori. Otherwise, we need the following trick. Consider the graph $X \subset \mathbb{T}^{r+s}$ of the map f with coordinates $(t_1, \ldots, t_r, y_1, \ldots, y_s)$ defined by equations $f_1(t) - y_1 = \cdots = f_s(t) - y_s = 0$. Let $\alpha : \mathbb{T}^{r+s} \to \mathbb{T}^s$ be the projection $(t, y) \mapsto y$. Then, $Y = \alpha(X)$ and α is a homomorphism of tori so we may apply Theorem 1 to compute $\mathcal{T}(Y)$. Thus, it remains for us to compute $\mathcal{T}(X)$. We can accomplished this using the following divisorial characterization of tropical varieties [3, §2], also known as geometric tropicalization.

Theorem 2 (Geometric Tropicalization). *Let $\mathbb{T}^n$ be the n-dimensional tori over $\mathbb{C}$ with coordinate functions $t_1, \ldots, t_n$, and let X be a closed subvariety. Assume that X is smooth and $\overline{X} \supset X$ is any compactification whose boundary $D = \overline{X} \setminus X$ is a divisor with simple normal crossings. Let $D_1, \ldots, D_m$ denote the irreducible components of D, and write $\Delta_{X,D}$ for the simplicial complex on $\{1, \ldots, m\}$ with $\{i_1, \ldots, i_l\} \in \Delta_{X,D}$ if and only if $D_{i_1} \cap \cdots \cap D_{i_l}$ is non-empty. Define the vector $[D_i] = (\mathrm{val}_{D_i}(t_1), \ldots, \mathrm{val}_{D_i}(t_n)) \in \mathbb{Q}^n$ where $\mathrm{val}_{D_i}(t_j)$ is the order of zero-poles of t_j along D_i. For any $\sigma \in \Delta_{X,D}$, let $[\sigma]$ be the cone in $\mathbb{Q}^n$ spanned by $\{[D_i] : i \in \sigma\}$. Then,*

$$\mathcal{T}(X) = \bigcup_{\sigma \in \Delta_{X,D}} [\sigma].$$

In order to compute $\mathcal{T}(X)$ using the above theorem, we need to produce a compactification $\overline{X} \supset X$ whose boundary has simple normal crossings. One way is to first compactify X in $\mathbb{P}^n \supset \mathbb{T}^n$ to get $\widetilde{X}$, and then find a resolution of singularities for the boundary $\widetilde{X} \setminus X$. Finding such a resolution can often be difficult. In [5], it was suggested that a boundary D with *combinatorial* normal crossings might suffice for geometric tropicalization to work, so a search for a resolution of singularities may not be necessary.

Combining Theorem 1 and 2, we get the following algorithm for computing $\mathcal{T}(Y)$ in our implicitization problem. We will demonstrate this algorithm by an example in Section 3.

Algorithm 3 (Tropical Implicitization).

Input: A rational map $f = (f_1, \ldots, f_s) : \mathbb{T}^r \to \mathbb{T}^s$, $f_i \in \mathbb{C}[t_1^{\pm}, \ldots, t_r^{\pm}]$.
Output: The set $\mathcal{T}(Y) \subset \mathbb{Q}^s$, $Y = \overline{\mathrm{Im} f}$.

1. Let $X = \mathbb{T}^r \setminus \cup_{i=1}^s \{f_i = 0\}$ so f induces a morphism $X \to Y \subset \mathbb{T}^s$.

2. Find a compactification $\overline{X} \supset X$ whose boundary D has simple normal crossings. Let $D_1, \ldots, D_m$ be the irreducible components of D.

3. For each D_i, compute $[D_i] = (\mathrm{val}_{D_i}(f_1), \ldots, \mathrm{val}_{D_i}(f_s)) \in \mathbb{Q}^s$.

4. Compute the simplicial complex $\Delta_{X,D}$ defined in Theorem 2. For each face $\sigma \in \Delta_{X,D}$, define $[\sigma]$ to be the cone spanned by $\{[D_i] : i \in \sigma\}$.

5. Return $\mathcal{T}(Y) = \bigcup_{\sigma \in \Delta_{X,D}} [\sigma]$.

For the case $r = 2$, the authors of [5] propose a suitable compactification which is constructed in two steps. First, they consider the toric surface $\mathbb{P}(\mathcal{N})$ corresponding to the fan $\mathcal{N}$ which is a strictly simplicial refinement of the normal fan of the Minkowski sum of the Newton polytopes of $f_1, \ldots, f_s$. Then, $\mathbb{P}(\mathcal{N}) \supset X$ is a compactification of X whose boundary $\mathbb{P}(\mathcal{N}) \setminus X$ consists of strict transforms E_i of (f_i) and toric divisors D_j associated to the rays ρ_j of $\mathcal{N}$. Furthermore, each divisor (f_i) can be written as

$$(f_i) = E_i + \sum_j \Psi_i(\rho_j) D_j$$

where $\Psi = (\Psi_1, \ldots, \Psi_s)$ is the tropicalization of the map $f = (f_1, \ldots, f_s)$.

At this stage, the divisors E_i and D_j may not have combinatorial normal crossings, i.e. no three divisors intersect at a point. The second step is then to perform a series of blow-ups with smooth centers to achieve this normal crossing condition, giving a morphism $\overline{X} \hookleftarrow \mathbb{P}(\mathcal{N})$. The boundary $\overline{X} \setminus X$ will then be made up of the proper transforms $\tilde{E}_i$, $\tilde{D}_j$ and the exceptional divisors F_k of the blow-ups. Additionally, on $\overline{X}$ we have

$$(f_i) = \tilde{E}_i + \sum_j \Psi_i(\rho_j) \tilde{D}_j + \sum_k (u_{ki} + \sum_j \Psi_i(\rho_j) v_{kj}) F_k$$

where u_{ki}, v_{kj} are the coefficients of F_k in the pullback of E_i and D_j respectively. Thus, this gives us rays in $\mathcal{T}(Y)$ generated by vectors

$$\begin{aligned} [\tilde{E}_i] &= e_i \\ [\tilde{D}_j] &= \Psi(\rho_j) \\ [F_k] &= (u_{k1}, \ldots, u_{ks}) + \Psi(\textstyle\sum_j v_{kj} \rho_j) \end{aligned} \tag{1}$$

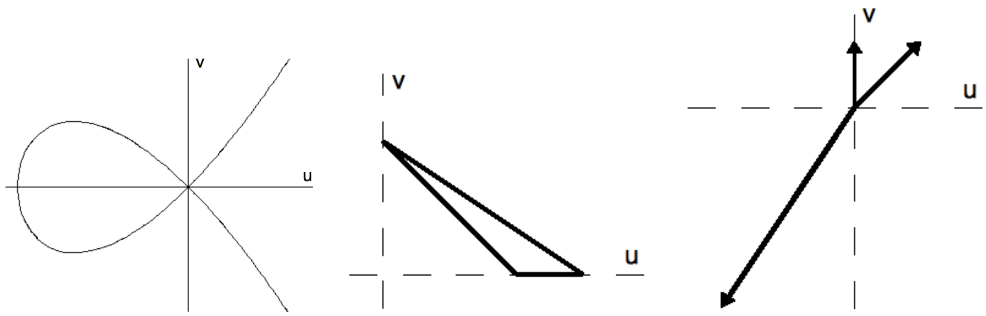
where $e_1, \ldots, e_s$ are the standard basis vectors of $\mathbb{Q}^s$. We demonstrate this $r = 2$ case by explaining in Section 4 the significance of Figure 1.

3 Example: The Nodal Cubic

Consider the nodal cubic curve

$$Y = \{(u, v) \in \mathbb{T}^2 : v^2 - u^3 - u^2 = 0\}.$$

Computing the tropicalization $\mathcal{T}(Y)$ is easy: it is the union of non-maximal cones in the normal fan of the Newton polytope of $v^2 - u^3 - u^2$ so it consists of three rays generated by $(0, 1)$, $(1, 1)$ and $(-2, -3)$.



(a) Nodal cubic (b) Newton polytope (c) Tropical variety $\mathcal{T}(Y)$

Figure 2: Tropicalization of the nodal cubic

We now compute $\mathcal{T}(Y)$ using geometric tropicalization. First, note that Y is the Zariski closure of the rational map

$$f : \mathbb{T}^1 \to \mathbb{T}^2, t \mapsto (f_1, f_2) = (t^2 - 1, t^3 - t).$$

Applying Algorithm 3, we have $X = \mathbb{T}^1 \setminus \{-1, 1\}$. This set can be compactified by $\mathbb{P}^1 \supset X$. The boundary $\mathbb{P}^1 \setminus X$ consists of four points $\{0, 1, -1, \infty\}$. Therefore, the corresponding divisors are $D_1 = (t)$, $D_2 = (t-1)$, $D_3 = (t+1)$ and $D_4 = (\frac{1}{t})$. Writing the divisors (f_1) and (f_2) in terms of these D_i,

$$\begin{array}{lllllll} (f_1) & = ((t+1)(t-1)) & = & & D_2 & +D_3 & -2D_4 \\ (f_2) & = (t(t+1)(t-1)) & = & D_1 & +D_2 & +D_3 & -3D_4. \end{array}$$

Hence, the vectors $[D_i] \in \mathbb{Q}^2$ have coordinates

$$\begin{array}{lcl} [D_1] & = & (0, 1) \\ [D_2] & = & (1, 1) \\ [D_3] & = & (1, 1) \\ [D_4] & = & (-2, -3) \end{array}$$

Finally, since the simplicial complex

$$\Delta_{X,D} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$$

has four distinct points, the tropical variety $\mathcal{T}(Y)$ consists of rays generated by $(0, 1)$, $(1, 1)$ and $(-2, -3)$. In fact, this computation also shows that the multiplicity of regular points on the ray $(1, 1)$ is 2.

4 Example: The Hankel Matrix

In this section, we will primarily be interested in studying the image of the rational map $f = (f_0, f_1, \ldots, f_6) : \mathbb{T}^6 \to \mathbb{T}^7$ where the

$$f_i = a_0 b_0^i b_1^{6-i} + a_1 c_0^i c_1^{6-i}$$

are polynomials in the Laurent polynomial ring $\mathbb{C}[a_0^\pm, a_1^\pm, b_0^\pm, b_1^\pm, c_0^\pm, c_1^\pm]$. The closure Y of the image is the first secant of a Veronese variety. Its ideal I_Y is generated by 3×3 minors of the Hankel matrix [4, Ex 1.2]

$$\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & y_6 \end{pmatrix}.$$

We feed the generators of I_Y into `Gfan` to compute $\mathcal{T}(Y) \subset \mathbb{Q}^7$. Because `Gfan` uses the max-plus convention, we negate the output vectors and present them in the min-plus convection in this section. The tropical variety is 4-dimensional with a 2-dimensional linearity space V spanned by

$$\{(0, 1, 2, 3, 4, 5, 6), (1, 1, 1, 1, 1, 1)\}.$$

Modulo this linearity space, $\mathcal{T}(Y)$ has 16 rays and 28 two-dimensional cones. The rays have coordinates

$$\begin{array}{lcl} v_0 & = & (1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) \\ v_6 & = & (0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0) \\ v_9 & = & (0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0) \\ v_{11} & = & (0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0) \\ v_{10} & = & (0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0) \\ v_7 & = & (0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0) \\ v_1 & = & (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1) \end{array}$$

$$\begin{array}{lcl} v_2 & = & (0,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1) \\ v_4 & = & (0,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2) \\ v_8 & = & (0,\ 1,\ 2,\ 3,\ 3,\ 3,\ 3) \\ v_5 & = & (0,\ 1,\ 2,\ 3,\ 4,\ 4,\ 4) \\ v_3 & = & (0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 5) \end{array}$$

$$\begin{array}{lcl} v_{13} & = & (1, \ 0, \ 1, \ 0, \ 1, \ 0, \ 1) \\ v_{14} & = & (0, \ 1, \ 0, \ 1, \ 0, \ 1, \ 0) \\ v_{12} & = & (1, \ 0, \ 0, \ 1, \ 0, \ 0, \ 1) \\ \\ v_{15} & = & (0, \ 1, \ 0, \ 3, \ 0, \ 1, \ 0) \end{array}$$

and are represented in Figure 1 by vertices with appropriate labels. A two-dimensional cone spanned by rays v_i and v_j is represented in Figure 1 by an edge between vertex i and j.

We now explain Figure 1 using geometric tropicalization. First, because

$$f_i(a_0, a_1, b_0, b_1, c_0, c_1) = a_0 b_0^i b_1^{6-i} + a_1 c_0^i c_1^{6-i} = f_i(a_0 b_1^{6-i}, a_1 c_1^{6-i}, b_0, 1, c_0, 1),$$

the image of f is the same as that of $g = (g_0, g_1, \ldots, g_6) : \mathbb{T}^4 \to \mathbb{T}^7$ where

$$g_i(a_0, a_1, b_0, c_0) = a_0 b_0^i + a_1 c_0^i.$$

This explains why Y and thus $\mathcal{T}(Y)$ is 4-dimensional. Next, we substitute $a_0 = a\lambda, a_1 = \lambda, b_0 = \omega, c_0 = c\omega$ to get

$$g_i(\lambda, \omega, a, c) = \lambda \omega^i (a + c^i).$$

This explains the 2-dimensional linearity space of $\mathcal{T}(Y)$. Therefore, it suffices to compute the tropicalization $\mathcal{T}(Z)$ of the closure Z of the image of the map $h = (h_0, h_1, \ldots, h_6) : \mathbb{T}^2 \to \mathbb{T}^7$ where $h_i(a, c) = a + c^i$. In particular, both $\mathcal{T}(Y)$ and $\mathcal{T}(Z)$ have the same projections onto the space orthogonal to V.

We compute $\mathcal{T}(Z)$ using Algorithm 3 and the proposed compactification for $r = 2$. Let $X = \mathbb{T}^2 \setminus \cup_{i=0}^6 \{f_i = 0\}$. The plots of the $\{f_i = 0\}$ in Figure 3 show that they do not have combinatorial normal crossings, so we need to blow up the crossing points. We do this for the origin by compactifying X with the toric surface $\mathbb{P}(\mathcal{N})$ where $\mathcal{N}$ is the normal fan of the Minkowski sum of the Newton polytopes of $h_0, \ldots, h_6$, as illustrated in Figure 4. The rays are denoted $\pm\rho_0, \pm\rho_1, \ldots, \pm\rho_6$. Let D_j and $\overline{D}_j$ be toric divisors corresponding to ρ_j and $-\rho_j$ respectively. At this stage, we can compute the following rays on $\mathcal{T}(Z)$ and $\mathcal{T}(Y)$, modulo the linear space V. Here, $\Psi = (\Psi_0, \ldots, \Psi_6)$ is the tropicalization of h, i.e. $\Psi_i(a, c) = \min(a, ic)$.

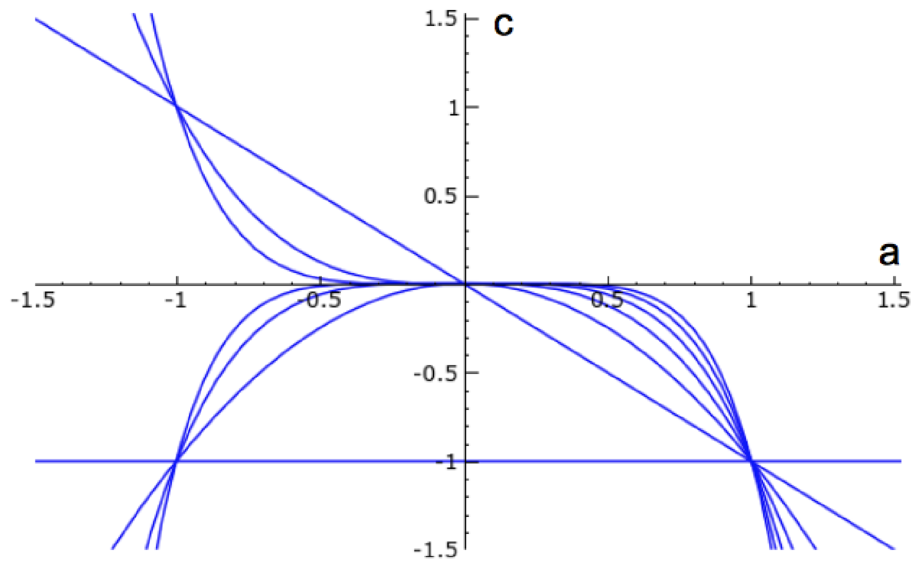


Figure 3: Plots of the $\{f_i = 0\}$

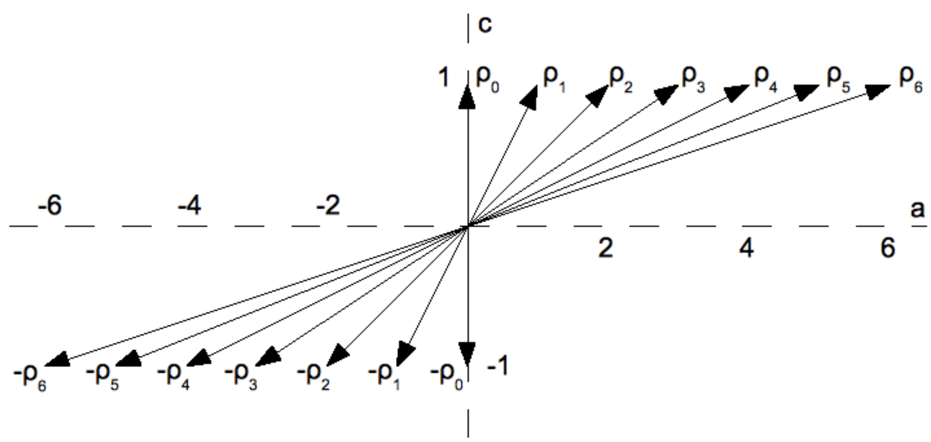


Figure 4: Normal fan $\mathcal{N}$

$$
\begin{array}{rcll}
[E_0] & = & (1,0,0,0,0,0,0) & \sim & v_0 \\
[E_1] & = & (0,1,0,0,0,0,0) & \sim & v_6 \\
[E_2] & = & (0,0,1,0,0,0,0) & \sim & v_9 \\
[E_3] & = & (0,0,0,1,0,0,0) & \sim & v_{11} \\
[E_4] & = & (0,0,0,0,1,0,0) & \sim & v_{10} \\
[E_5] & = & (0,0,0,0,0,1,0) & \sim & v_7 \\
[E_6] & = & (0,0,0,0,0,0,1) & \sim & v_1
\end{array}
$$

$$
\begin{array}{rcccccc}
[D_0] & = & \Psi(\rho_0) & = & (0,0,0,0,0,0,0) & \sim & 0 \\
[D_1] & = & \Psi(\rho_1) & = & (0,1,1,1,1,1,1) & \sim & v_2 \\
[D_2] & = & \Psi(\rho_2) & = & (0,1,2,2,2,2,2) & \sim & v_4 \\
[D_3] & = & \Psi(\rho_3) & = & (0,1,2,3,3,3,3) & \sim & v_8 \\
[D_4] & = & \Psi(\rho_4) & = & (0,1,2,3,4,4,4) & \sim & v_5 \\
[D_5] & = & \Psi(\rho_5) & = & (0,1,2,3,4,5,5) & \sim & v_3 \\
[D_6] & = & \Psi(\rho_6) & = & (0,1,2,3,4,5,6) & \sim & 0
\end{array}
$$

$$
\begin{array}{rcccccc}
[\overline{D}_0] & = & \Psi(-\rho_0) & = & (\ \ 0,-1,-2,-3,-4,-5,-6) & \sim & 0 \\
[\overline{D}_1] & = & \Psi(-\rho_1) & = & (-1,-1,-2,-3,-4,-5,-6) & \sim & v_2 \\
[\overline{D}_2] & = & \Psi(-\rho_2) & = & (-2,-2,-2,-3,-4,-5,-6) & \sim & v_4 \\
[\overline{D}_3] & = & \Psi(-\rho_3) & = & (-3,-3,-3,-3,-4,-5,-6) & \sim & v_8 \\
[\overline{D}_4] & = & \Psi(-\rho_4) & = & (-4,-4,-4,-4,-4,-5,-6) & \sim & v_5 \\
[\overline{D}_5] & = & \Psi(-\rho_5) & = & (-5,-5,-5,-5,-5,-5,-6) & \sim & v_3 \\
[\overline{D}_6] & = & \Psi(-\rho_6) & = & (-6,-6,-6,-6,-6,-6,-6) & \sim & 0
\end{array}
$$

Next, we find crossings between these divisors. First, observe that E_i intersects D_i simply and normally. Indeed, in the open subset of $\mathbb{P}(\mathcal{N})$ corresponding to the cone spanned by $\rho_j = (j,1)$ and $\rho_{j+1} = (j+1,1)$, $0 \leq j \leq 5$, the total transform of (f_i) comes from setting $a = u^j v^{j+1}, c = uv$:

$$f_i(u,v) = u^j v^{j+1} + (uv)^i.$$

Thus, the strict transform of (f_j) is $E_j = (v+1)$ and it intersects $D_j = (u)$. Similarly, $E_{j+1} = (1+u)$ intersects $D_{j+1} = (v)$, and computations on other cones show that E_i intersects $\overline{D}_i$. These crossings remain unchanged under blow-ups at other points on $\mathbb{P}(\mathcal{N})$ so they show up as 2-dimensional cones on $\mathcal{T}(Y)$. We indicate them by dashed edges in Figure 5. There are no other crossings between the E_i and the $D_j, \overline{D}_j$.

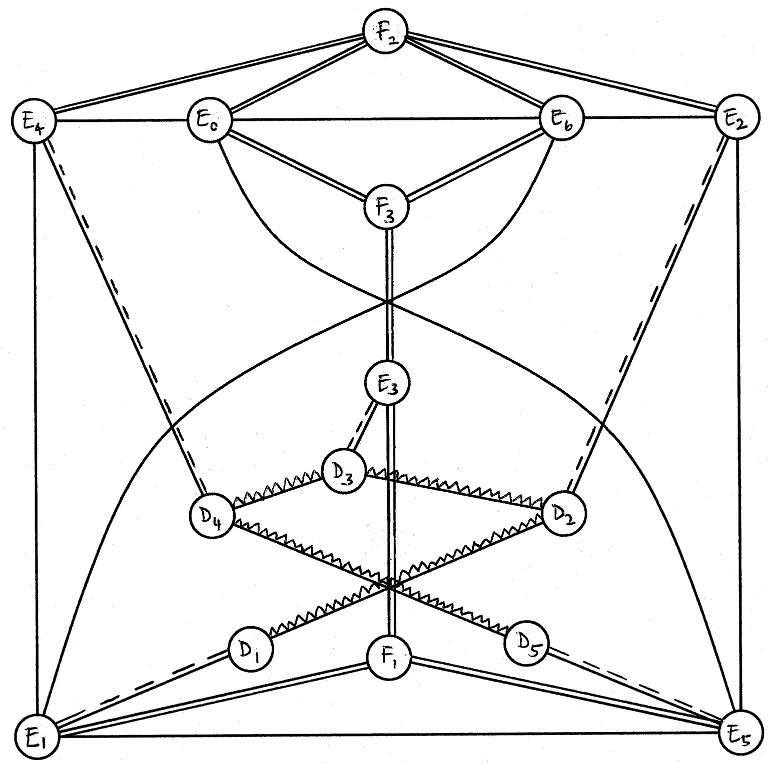


Figure 5: The tropical variety $\mathcal{T}(Y)$ annotated

As for crossings among the toric divisors D_j, $\overline{D}_j$ on $\mathbb{P}(\mathcal{N})$, two of them intersect if and only if their corresponding rays on $\mathcal{N}$ are adjacent. These crossings are indicated on Figure 5 by jagged edges.

Finally, we come to crossings among the strict transforms E_i of the (f_i). These crossings come from intersections of the curves $\{f_i(a,c) = 0\}$ in $\mathbb{T}^2$. If (a,c) is a crossing point of $f_i = a + c^i = 0$ and $f_j = a + c^j = 0$, then $c^i = -a = c^j$ so c is a k-th root of unity for $k \leq 6$. We tabulate powers of various roots of unity below, where ω, i and ζ are primitive third, fourth and fifth roots of unity.

	1	-1	ω	ω^2	i	$-i$	ζ	$-\omega$
1	1	1	1	1	1	1	1	1
c	1	-1	ω	ω^2	i	$-i$	ζ	$-\omega$
c^2	1	1	w^2	w	-1	-1	ζ^2	ω^2
c^3	1	-1	1	1	$-i$	i	ζ^3	-1
c^4	1	1	ω	ω^2	1	1	ζ^4	ω
c^5	1	-1	ω^2	ω	i	$-i$	1	$-\omega^2$
c^6	1	1	1	1	-1	-1	ζ	1

From each column, we can read off coordinates of the crossings. Indeed, the first column shows that crossing point $(a, c) = (-1, 1)$ has all seven curves $\{f_i = 0\}$ passing through it. Blowing up this point introduces an exceptional divisor F_0. The coefficient of F_0 in the pullback of each E_i is 1 and in the pullback of each $D_j, \overline{D}_j$, it is 0. Thus, by (1) this gives rise to the ray

$$[F_0] = (1, 1, 1, 1, 1, 1, 1) \sim 0.$$

The second column shows that $(1, -1)$ is a crossing point for $\{f_1, f_3, f_5\}$ and $(-1, -1)$ is one for $\{f_0, f_2, f_4, f_6\}$. Blowing up at these points produces exceptional divisors F_1 and F_2 respectively. They give rise to rays

$$\begin{aligned} [F_1] &= (0, 1, 0, 1, 0, 1, 0) \sim v_{14} \\ [F_2] &= (1, 0, 1, 0, 1, 0, 1) \sim v_{13}. \end{aligned}$$

Because of the blow-up, F_1 intersects the strict transforms of E_1, E_3, E_5, while F_2 intersects those of E_0, E_2, E_4, E_6. These intersections are represented by the double-lined edges in Figure 5.

The third and fourth column shows that $(-1, \omega)$ and $(-1, \omega^2)$ are crossing points for $\{f_0, f_3, f_6\}$. The exceptional divisors F_3 and $\overline{F}_3$ coming from the blow-up of these points both give rise to the ray

$$[F_3] = [\overline{F}_3] = (1, 0, 0, 1, 0, 0, 1) \sim v_{12}.$$

Their intersections with E_0, E_3, E_6 are also represented by double-lined edges.

The above are all crossing points with three or more curves intersecting. The remaining 13 crossings are simple normal crossings so we do not need to blow them up. They are represented by 8 single-lined edges in Figure 5. For example, the crossing $(-\zeta, \zeta)$ gives us the edge between E_1 and E_6. Finally, we note that the ray v_{15} lies on the cone spanned by v_{11} and v_{14} so it is extraneous. This completes our explanation of Figure 1.

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