Asymptotics of Laplace Integrals, and What is a Blow-up?

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(work-in-progress with Robin Pemantle)

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Problem

$f : [-1,1]^d \to \mathbb{R}$ a real analytic function
i.e. power series $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{R}[[x_1, \ldots, x_d]]$ convergent in $[-1,1]^d$
$\tau \in \mathbb{N}^d$ vector of nonnegative integers

Want to find asymptotics of Laplace integral

$$Z(n) \; = \; \int_{[-1,1]^d} e^{-n|f(\omega)|} |\omega^\tau| d\omega, \quad n \to \infty$$

(Varchenko, 1970) Laplace integrals have asymptotic expansion

$$\sum_{\alpha,i} c_{\alpha,i} \, n^{-\alpha} (\log n)^{i-1}$$

We are interested in first term $c_{\lambda,\theta} \, n^{-\lambda} (\log n)^{\theta-1}$.
(λ, θ) is the *real log canonical threshold* (RLCT) of f.

[I will now give a theorem, and define the terms later.]

[2] **Theorem (L.)** Let $f, \tau, Z(n), \lambda, \theta, c_{\lambda,\theta}$ be as before.

If f is nondegenerate, then $(\lambda, \theta) = (1/l_\tau, \theta_\tau)$ where l_τ is the τ-distance, and θ_τ its multiplicity, of the Newton polyhedron $\mathcal{P}(f)$.

Moreover, if $\mathcal{F}$ is a unimodular refinement of the normal fan $\mathcal{N}(f)$ and σ is the τ-cone of $\mathcal{N}_\tau(f)$, then

$$c_{\lambda,\theta} = \frac{2^\theta \lambda^\theta \, \Gamma(\lambda)}{(\theta - 1)!} \sum_{v \in \mathcal{F}_\sigma} \frac{1}{\prod_{i=1}^{\theta} \hat{\alpha}_i} \int_{[-1,1]^{d-\theta}} \frac{1}{|\hat{\pi}_v^* f(0, \bar{\mu})|^\lambda} |\bar{\mu}|^{\bar{\alpha}-1} d\bar{\mu}.$$

where α is the vector of row sums of v and $\hat{\pi}_v^* f(\mu) = \hat{\mu}^{-\hat{\alpha}/\lambda} f(\mu^v)$.

In particular, this theorem holds for $f = f_1^2 + \cdots + f_k^2$
when $\langle f_1, \ldots, f_k \rangle$ is a monomial ideal.

Crash Course on Polyhedral Fans

Let $\sigma \subset \mathbb{R}^d$ be a cone
polyhedral: generated by $v_1, \ldots, v_k \in \mathbb{R}^d$, i.e. $\sigma = \{\sum_i \lambda_i v_i : \lambda_i \geq 0\}$
rational: generated by $v_1, \ldots, v_k \in \mathbb{Z}^d$
pointed: origin is a face
ray: one-dimensional and pointed
min gen of a rational ray: the generator in $\mathbb{Z}^d$ of minimal length
min gens of a pointed rational polyhedral cone: the min gens of its edges
smooth: min gens generate all lattice points $\sigma \cap \mathbb{Z}^d$ over $\mathbb{Z}$
simplicial: min gens linearly independent over $\mathbb{R}^d$
unimodular: smooth and simplicial
(min gens form rows of matrix $v \in \mathbb{Z}^{d \times d}$ with det ± 1)

Let $\mathcal{F}$ be a collection of pointed rational polyhedral cones in $\mathbb{R}^d$
smooth, simplicial, unimodular: every cone is ditto
fan: 1. every face of every cone is in $\mathcal{F}$
2. intersection of every two cones is in $\mathcal{F}$
support: union of every cone as subset of $\mathbb{R}^d$
locally complete: support is nonnegative orthant $\mathbb{R}^d_{\geq 0}$
$\mathcal{F}_1$ *refinement* of $\mathcal{F}_2$: cones of $\mathcal{F}_1$ partition cones of $\mathcal{F}_2$

Given locally complete fan $\mathcal{N}$ in $\mathbb{R}^d$,
cone $\sigma \in \mathcal{N}$ of dimension θ,
unimodular refinement $\mathcal{F}$ of $\mathcal{N}$,
$\mathcal{F}_\sigma$: set of maximal cones of $\mathcal{F}$ which intersect σ in dimension θ.
Represent each maximal cone by matrix $v \in \mathbb{Z}^{d\times d}$ of min gens
where first θ rows lie in σ, forming submatrix $\hat{v} \in \mathbb{Z}^{\theta\times d}$.
Write $\mu = (\hat{\mu}, \bar{\mu}) \in \mathbb{R}^\theta \times \mathbb{R}^{d-\theta}$ for $\mu \in \mathbb{R}^d$.

Given unimodular matrix $v \in \mathbb{Z}^{d\times d}$,
monomial map $\pi_v : \mathbb{R}^d \to \mathbb{R}^d, \mu \mapsto \mu^v = (\mu^{v_{\cdot 1}}, \ldots, \mu^{v_{\cdot d}})$.
(min gens form rows of v but π_v is given by columns of v)
Given power series $f(x) \in \mathbb{R}[[x_1, \ldots, x_d]]$,
pullback $\pi_v^* f(\mu) = f \circ \pi_v(\mu) = f(\mu^v) \in \mathbb{R}[[\mu_1, \ldots, \mu_d]]$

Newton Polyhedron and Nondegeneracy

Given polyhedron $\mathcal{P} \subset \mathbb{R}^d_{\geq 0}$ whose normal fan $\mathcal{N}$ is locally complete,
vector $\tau \in \mathbb{N}^d$ of nonnegative integers
consider the ray generated by $(\tau + 1) = (\tau_1 + 1, \ldots, \tau_d + 1)$
τ-distance l_τ: smallest t such that $t(\tau + 1) \in \mathcal{P}$
τ-face $\mathcal{P}_\tau$: smallest face of $\mathcal{P}$ containing $l_\tau(\tau + 1)$
τ-cone $\mathcal{N}_\tau$: cone of $\mathcal{N}$ corresponding to $\mathcal{P}_\tau$
multiplicity θ_τ of l_τ: dimension of $\mathcal{N}_\tau$

[Note: Translating $\mathcal{P}$ changes $\mathcal{N}_\tau$ so it is not largest cone containing $(\tau + 1)$]

Given power series $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{R}[[x_1, \ldots, x_d]]$,
Newton polyhedron $\mathcal{P}(f) = \text{conv}\{\alpha + \mathbb{R}^d_{\geq 0} : c_\alpha \neq 0\}$
face polynomial $f_\gamma = \sum_{\alpha \in \gamma} c_\alpha x^\alpha$, for each face $\gamma \subset \mathcal{P}(f)$
A real analytic function f is *nondegenerate* iff for all compact faces $\gamma \subset \mathcal{P}(f)$,
f_γ is nonsingular in the torus $(\mathbb{R}^*)^d$
(f_γ is *not* in the γ-discriminantal variety) [Felipe]

$$\text{i.e. } f_\gamma = \frac{\partial f_\gamma}{\partial x_1} = \cdots = \frac{\partial f_\gamma}{\partial x_d} = 0 \text{ has no roots in } (\mathbb{R}^*)^d$$

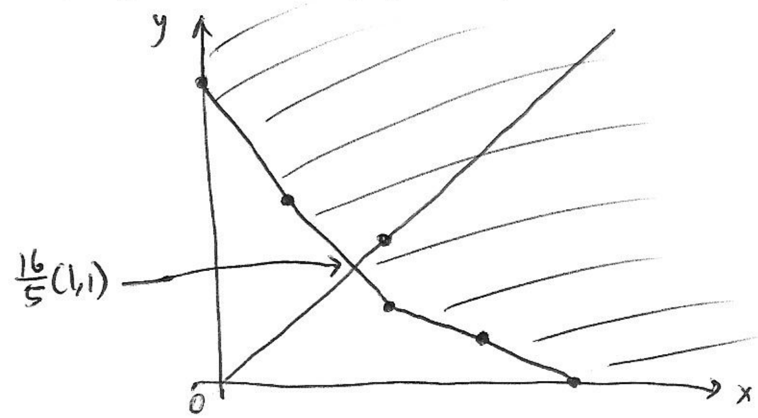
Example

Find the asymptotics of

$$Z(n) = \int_{-1}^{1}\int_{-1}^{1} e^{-n|f(x,y)|}dxdy$$

$$f(x,y) = x^8 + x^6 y + x^4 y^4 + x^4 y^2 + x^2 y^5 + y^8, \quad \tau = (0,0)$$

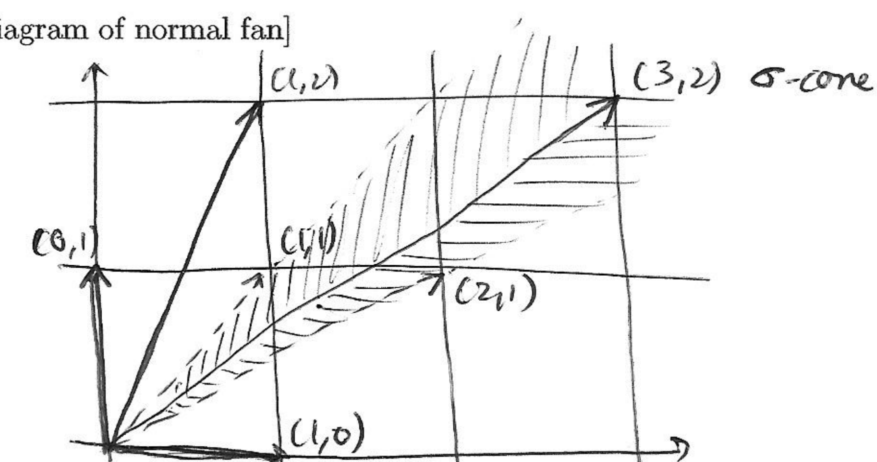
[Diagram of Newton polyhedron]



[Draw $t(1,1)$ diag, τ-face $3x + 2y = 16$, τ-dist $16/5$, mult 1]

RLCT $(\lambda, \theta) = (5/16, 1)$, asymptotics $Z(n) \approx cn^{-5/16}$

[Diagram of normal fan]



[Use dotted line to draw refinement, show τ-cone σ, shade $\mathcal{F}_\sigma$]

$$\mathcal{F}_\sigma = \left\{ \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

$$v = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \qquad f(a^3b, a^2b) = a^{16}b^6(1 + b + b^2 + a^8b^2 + a^4v + a^4b^2)$$

$$\hat{\pi}_v^* f(0, b) = b^6(1 + b + b^2)$$

$$c = \frac{2 \cdot \frac{5}{16} \cdot \Gamma(\frac{5}{16})}{0!} \left(\frac{1}{5} \int_{-1}^{1} \frac{b}{(b^6(1 + b + b^2))^{5/16}} db + \frac{1}{5} \int_{-1}^{1} \frac{b^2}{(b^8(1 + b + b^2))^{5/16}} db \right)$$

$$= 7.0781612919 \ (\texttt{Maple})$$

$n = 10^k$	actual	estimate	estimate/actual
1	1.854933489	3.446834688	1.858198533
2	1.007139893	1.678496558	1.666597232
3	0.5360445536	0.8173733149	1.524823467
4	0.2799653636	0.3980342602	1.421726799
5	0.1441396519	0.1938297586	1.344735859
6	0.07344558094	0.09438879785	1.285152853
7	0.03713263466	0.04596427930	1.237840507
8	0.01865838290	0.02238311133	1.199627612
9	0.009328949800	0.01089984833	1.168389643
10	0.004645366734	0.005307872167	1.142616390

Incidentally, $2f(x, y) = f_1^2 + f_2^2 + f_3^2 = (x^4 + x^2y)^2 + (x^4 + y^4)^2 + (y^4 + x^2y)^2$, where $\langle f_1, f_2, f_3 \rangle = \langle x^4, y^4, x^2y \rangle$. Thus, f is nondegenerate.

Proof Idea:

1. Asymptotics of Laplace integral $\leftrightarrow$ Poles of zeta function (Arnold et al.)

$$\zeta(z) = \int_{[-1,1]^d} |f(\omega)|^{-z} |\omega^\tau| d\omega$$

   The RLCT (λ, θ) is the smallest pole and its multiplicity of $\zeta(n)$, and

$$c_{\lambda,\theta} = \frac{\Gamma(\lambda)}{(\theta-1)!} d_{\lambda,\theta}$$

   where $d_{\lambda,\theta}$ is coefficient of $(\lambda - z)^{-\theta}$ in Laurent expansion of $\zeta(n)$.

2. If f is nondegenerate, then $\pi : \mathbb{P}(\mathcal{N}(f)) \to \mathbb{R}^d$ resolves f (Varchenko)

What is a Blow-up?

1. Blow-up of the origin

Consider $\mathbb{R}^n \times \mathbb{P}^{n-1}$ with coords $((x_1, x_2, \ldots, x_n), (\xi_1 : \xi_2 : \ldots : \xi_n))$.
Let V be subset of points $((x_1, x_2, \ldots, x_n), (x_1 : x_2 : \ldots : x_n))$, $x \in \mathbb{R}^n \setminus \{0\}$.
Then, $X = \overline{V} \subset \mathbb{R}^d \times \mathbb{P}^{n-1}$ is the blow-up of the origin in $\mathbb{R}^n$.
The projection $\pi : X \to \mathbb{R}^n, (x, \xi) \mapsto x$, is the blow-up map.

1. $X = V \cup E$ where $E = \{0\} \times \mathbb{P}^{n-1}$ is the exceptional divisor.
   π is an isomorphism from V to $\mathbb{R}^n \setminus \{0\}$, while $\pi^{-1}(0) = E$.

2. X is a toric variety (defined by binomials).

$$X = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{P}^{n-1} \mid x_i \xi_j = x_j \xi_i \text{ for all } i, j\}$$

3. X covered by affine charts $U_i = \{(x, \xi) \in X \mid \xi_i \neq 0\} \simeq \mathbb{R}^n$ with coords

$$\{(y_1, \ldots, y_n) = (\frac{\xi_1}{\xi_i}, \ldots, x_i, \ldots, \frac{\xi_n}{\xi_i}\}$$

   so π is given by affine maps $\pi_i : U_i \to \mathbb{R}^n$

$$(y_1, \ldots, y_n) \mapsto (y_1 y_i, \ldots, y_i, \ldots, y_n y_i)$$

2. Blow-up of a linear subspace

$\pi \times id : X \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ is the blowing-up of $\{0\} \times \mathbb{R}^m$ in $\mathbb{R}^{n+m}$.

3. Blow-up of a smooth center C

Cover C with affine charts such that in each chart, C is a linear subspace of a nbd of the origin. Restrict blow-up maps to nbds. Glue maps together.

4. **Theorem** (Hironaka): Every variety is birational to a smooth variety via a sequence of blow-ups along smooth centers.

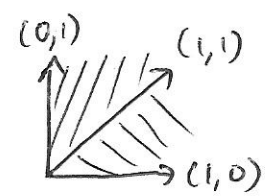
Toric Blow-ups

Given unimodular locally complete fan $\mathcal{F}$ in $\mathbb{R}^d$,
we have smooth toric variety $\mathbb{P}(\mathcal{F})$
covered with affine charts $U_\sigma \simeq \mathbb{R}^d$, one for each max cone $\sigma \in \mathcal{F}$.

We also have the toric blow-up $\pi_{\mathcal{F}} : \mathbb{P}(\mathcal{F}) \to \mathbb{R}^d$
defined on affine charts by monomial maps

$$\pi_v : U_\sigma \to \mathbb{R}^d, \quad \mu \mapsto \mu^v$$

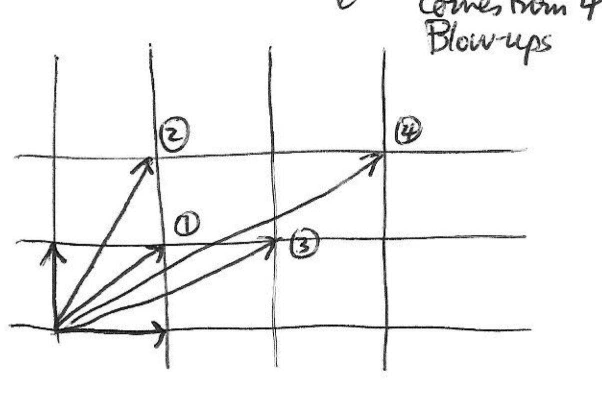
where min gens of σ form rows of the matrix v.
Every ray (except $e_1, \ldots, e_d$) gives an exceptional divisor.



e.g. Blow-up of a point in R^d comes from fan given by standard bases $e_1, \ldots, e_d$ and $e_0 = e_1 + \ldots + e_d$.

Conjecture: every toric blow-up is a seq of blow-ups along linear subspaces?
[Example from asymptotics]

Theorem (Varchenko) If f is nondegenerate, then the toric blow-up coming from its normal fan $\mathcal{N}(f)$ resolves f.



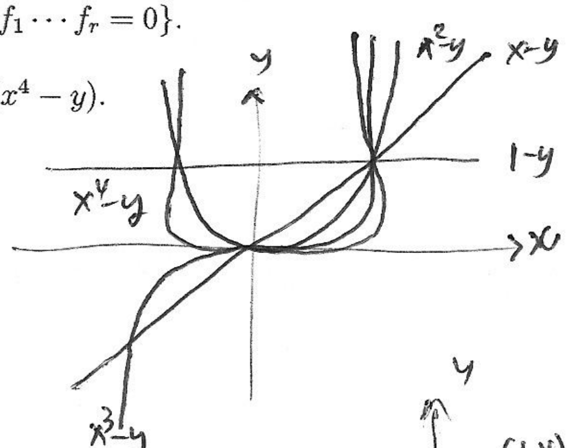
[End with this example, for Melody]
Example (Tropical Implicitization)

Let $\mathbb{T}^r = (\mathbb{C}^*)^r$ be the torus.
Given rational map $f = (f_1, \ldots, f_s) : \mathbb{T}^r \to \mathbb{T}^s$, $f_i \in \mathbb{C}[t_1^{\pm}, \ldots, t_r^{\pm}]$
Want tropicalization of the image Y of f.

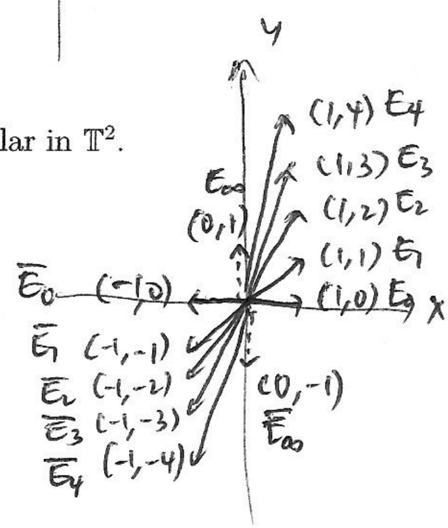
Let $X = \mathbb{T}^r \setminus \{f_1 \cdots f_r = 0\}$ so morphism $f : X \to Y$.
Need compactification $\bar{X}$ of X.

Method 1: Embed $X \hookrightarrow \mathbb{P}^r$, resolve singularities of boundary divisor $\mathbb{P}^r \setminus X$
Method 2: Resolve singularities of $\mathbb{T}^r \setminus X = \{f_1 \cdots f_r = 0\}$.

e.g. $f_1 \cdots f_r = (1-y)(x-y)(x^2-y)(x^3-y)(x^4-y)$.
[Picture of boundary divisors]



Normal fan is Minkowski sum of normal fans.
[Picture of normal fan]
Nondegenerate?
e.g. in dir $(1,2)$, face poly $(1)(x)(x^2-y)(-y)(-y)$ nonsingular in $\mathbb{T}^2$.



[Refinement $\mathcal{F}$ of normal fan]
Blow-up $\pi_{\mathcal{F}} : \mathbb{P}(\mathcal{F}) \to \mathbb{T}^r$.
Compute pullbacks of divisors.
Let $D = (x^2 - y)$. Let $\Phi(x,y) = \min(2x, y)$.
In chart $(1,1), (1,2)$, $x^2 - y = (uv)^2 - u^1v^2 = uv^2(u-1)$.
So E_1 has mult $1 = \Phi(1,1)$, E_2 has mult $2 = \Phi(1,2)$.

$\pi_{\mathcal{F}}^*(D) = D' + E_1 + 2E_2 + 2E_3 + 2E_4 - 2\bar{E}_0 - 2\bar{E}_1 - 2\bar{E}_2 - 3\bar{E}_3 - 4\bar{E}_4 - \bar{E}_\infty$.