Student Analysis Seminar: Asymptotics of Laplace Integrals in Statistics

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Abstract

Consider the Laplace integral $Z(n) = \int_\Omega \exp(-nf(\omega))\,\varphi(\omega)d\omega$ in a small neighborhood Ω of the origin in real space, where f and φ are real analytic functions. How do we derive the asymptotics of this integral as the integer n grows large? We outline the basics of asymptotic theory: Mellin transforms, zeta functions, resolution of singularities and real log canonical thresholds. When the amplitude f is nondegenerate, we may employ combinatorial tools such as Newton polyhedra and toric resolutions. The approximation of such integrals is important in statistics, and we explore some interesting mathematics arising from this context. If time permits, we will demonstrate software computations of asymptotic coefficients and higher order terms.

I realize I'm an algebraic geometer giving an analysis talk to expert analysts, so please forgive me and let me know if I make any mistakes. Also, let me know if you see any related ideas.

1 Introduction

Problem. Find asymptotic expansion of

$$Z(n) = \int_\Omega e^{-nf(\omega)}\,\varphi(\omega)\,d\omega, \quad n \to \infty.$$

$\Omega \subset \mathbb{R}^d$	
n	positive integer
$f(\omega)$	real analytic function
$\varphi(\omega)\,d\omega$	measure on Ω

Figure out other assumptions later.

What would such an expansion look like? Polynomial? Exponential?

Remark. Assume $f(\omega)$ bounded below over Ω.

Otherwise, difficult to ensure convergence of integral. If $f_0 = \inf_{\omega\in\Omega} f(\omega)$, then

$$Z(n) = (e^{-f_0})^n \int_\Omega e^{-n(f(\omega)-f_0)}\varphi(\omega)\,d\omega$$

so assume $\inf f = 0$.

2 Asymptotic Theory

Suppose the following function is well-defined.

$$v(t) = \frac{d}{dt} \int_{\omega \in \Omega:\, 0 < f(\omega) < t} \varphi(\omega) d\omega.$$

This is the Gelfand-Leray function or state density function.

Then, $Z(n)$ is the Laplace transform of $v(t)$.

$$Z(n) = \int_0^\infty e^{-nt}\, v(t) dt = \int_\Omega e^{-nf(\omega)}\, \varphi(\omega) d\omega$$

Now, consider the Mellin transform of $v(t)$ (the zeta function).

$$\zeta(z) = \int_0^\infty t^{-z}\, v(t) dt = \int_\Omega f(\omega)^{-z}\, \varphi(\omega) d\omega$$

Our key to unlocking the asymptotics of $Z(n)$ is $v(t)$.

Big Picture.

1. Assume Ω, $f(\omega)$ and $\varphi(\omega)d\omega$ has special structure.
2. Show $\zeta(z)$ has Laurent expansion

$$\zeta(z) = \sum_\alpha \sum_{i=1}^{d} d_{i,\alpha} (z - \alpha)^{-i}$$

3. Show $v(t)$ well-defined and has expansion (via inverse-Mellin transform of $(z - \alpha)^{-i}$)

$$v(t) = \sum_\alpha \sum_{i=1}^{d} b_{i,\alpha} t^\alpha (\log t)^i$$

4. Show $Z(n)$ has asymptotic expansion (via Laplace transform of $t^\alpha (\log t)^i$)

$$Z(n) = \sum_\alpha \sum_{i=1}^{d} c_{i,\alpha} n^{-\alpha} (\log n)^{i-1}$$

Corollaries.

1. $Z(n)$ is asymptotically proportional to $n^{-\lambda} (\log n)^{\theta - 1}$
   where λ smallest pole of $\zeta(z)$ and θ its multiplicity.
   Define real log canonical threshold $\text{RLCT}_\Omega(f; \varphi) = (\lambda, \theta)$.
2. The asymptotic coefficients $c_{i,\alpha}$ can be expressed in terms of the Laurent coefficients $d_{i,\alpha}$.

$$c_{i,\alpha} = \frac{(-1)^i}{(i-1)!} \sum_{j=i}^{d} \frac{\Gamma^{(j-i)}(\alpha)}{(j-i)!} d_{j,\alpha}$$

3 Resolution of Singularities

Find special structure on Ω, $f(\omega)$ and $\varphi(\omega)d\omega$ so that $\zeta(z)$ has Laurent expansion.

Lemma. Let $\Omega = [0,1]^d$, $f(\omega) = \omega^\kappa$ and $\varphi(\omega)d\omega = \omega^\tau d\omega$. Then, the zeta function

$$\zeta(z) = \int_{[0,1]^d} (\omega^\kappa)^{-z} \omega^\tau d\omega = \prod_{i=1}^{d} \frac{1}{-\kappa_i z + \tau_i + 1}$$

can be continued analytically to the whole complex plane, and has a Laurent expansion.

Improvements.

1. $\varphi = \omega^\tau \varphi_s$, φ_s positive smooth. (bound φ_s)
2. $\varphi = \varphi_a \varphi_s$, φ_a real analytic. (use Taylor expansion of φ_a)
3. Ω compact neighborhood of origin. (bound f away from origin)
4. f real analytic function. (use resolution of singularities of f)
5. Ω compact semianalytic set. (use resolution of singularities of $\partial\Omega$)
6. $\varphi = |\varphi_a \varphi_s|$. (use resolution of singularities of φ_a)

Remark. Global asymptotics = sum of local asymptotics.

Enough to study compact semialgebraic nbds Ω of the origin, with $f(\omega) \geq 0, f(0) = 0$.

4 Nondegeneracy

Class of functions which are easy to resolve.
Suppose $f = \sum_\alpha c_\alpha \omega^\alpha$ real analytic at origin.

Definition. Newton polyhedron $\mathcal{P}(f) = \text{conv}\{\alpha + \mathbb{R}_+^d : c_\alpha \neq 0\}$

Definition. Face polynomial $f_\gamma(\omega) = \sum_{\alpha\in\gamma} c_\alpha \omega^\alpha$, γ compact face of $\mathcal{P}(f)$.

Recall f is singular at $x \in \mathbb{R}^d$ if $f(x) = \frac{\partial f}{\partial \omega_1}(x) = \cdots = \frac{\partial f}{\partial \omega_2}(x) = 0$.

Definition. f nondegenerate at the origin w.r.t. its Newton polyhedron
if for all compact faces $\gamma \subset \mathcal{P}(f)$, f_γ is nonsingular in torus $(\mathbb{R}^*)^d$.

Definition. Given $\tau \in \mathbb{Z}_{\geq 0}^d$ and polyhedron $\mathcal{P}$, define
τ-distance l_τ = smallest $t \geq 0$ such that $t(\tau + 1) \in \mathcal{P}$.
τ-multiplicity θ_τ = codim of face of $\mathcal{P}$ at this intersection.

Theorem. If f real analytic at origin, Ω sufficiently small nbd of origin, $\tau \in \mathbb{Z}_{\geq 0}^d$,

$$\text{RLCT}_\Omega(f; \omega^\tau) \leq (1/l_\tau, \theta_\tau).$$

Equality holds when f is ndg.

The main idea is to resolve f using a toric map defined by the normal fan of $\mathcal{P}(f)$.

Example. $f = x^4 + x^2y + xy^3 + y^4$.
Newton polyhedron, face polynomial, τ-distance, τ-multiplicity,
normal fan, blow-up in one of the cones.

In fact, if f ndg, we can compute all higher order asymptotics of $Z(n)$.

Theorem. If f ndg, then

$$\int_{[0,1]^d} e^{-nf(\omega)} \omega^\tau d\omega \quad \approx \quad Cn^{-\lambda}(\log n)^{\theta-1}$$

$$C = \frac{\Gamma(\lambda)}{(\theta-1)!} \sum_{v\in\mathcal{F}} \prod_{i=1}^{\theta} (v\beta)_i^{-1} \int_{[0,1]^{d-\theta}} g_v(\bar{\mu}) d\bar{\mu}.$$

$(\lambda, \theta) = (1/l_\tau, \theta_\tau)$, $\bar{\mu} = (\mu_{\theta+1}, \ldots, \mu_d)$, $\bar{e} = e_{\theta+1} + \ldots + e_d$,
$g_v(\bar{\mu})$ is $f(\mu^v)^{-\lambda} \mu^{v\tau+\alpha-\bar{e}}$ evaluated at $(0, \ldots, 0, \bar{\mu})$,
$\mathcal{F}$ is a unimodular refinement of the normal fan of $\mathcal{P}(f)$,
$\beta(v)$ is the vertex of $\mathcal{P}(f)$ corresponding to v,
$\alpha(v)$ is the vector of row sums of v.

Similar formulas for higher order terms.

5 Statistics

Consider an event E with k outcomes, with probabilities $p_1, p_2, \ldots, p_k$.
Make n observations of E. Let $q_1, q_2, \ldots, q_k$ be relative frequences of each outcome.

$$\text{Prob of data} = p_1^{nq_1} p_2^{nq_2} \cdots p_k^{nq_k}$$

Let the $p_i(\omega)$ be functions of some parameter $\omega \in \Omega$, and $\varphi(\omega)$ be probability distribution on ω. This defines a model M. e.g. biased coin $p_1 = x, p_2 = 1 - x$.

$$\text{Prob of } M \text{ given data} = Z(n) = \int_\Omega \left(\prod_{i=1}^k p_i(\omega)^{q_i}\right)^n \varphi(\omega) d\omega$$

Assume $p(0) = q$ and Ω is sufficiently small nbd of origin.

$$Z(n) = \left(\prod_{i=1}^k q_i^{q_i}\right)^n \int_\Omega e^{-nK(\omega)} \varphi(\omega) d\omega$$

$$K(\omega) = \sum_{i=1}^k q_i \log \frac{q_i}{p_i(\omega)} \quad \text{Kullback-Leibler function}$$

Goal. Find $\mathrm{RLCT}_\Omega(K;\varphi)$.

How do we find a resolution of singularities for $K(\omega)$? We don't.

5.1 RLCT of ideals

Definition. $f \sim g$ over Ω if $\exists\, a, b > 0$ such that $af(\omega) \leq g(\omega) \leq bf(\omega)\ \forall \omega \in \Omega$.

Proposition. $f \sim g$ over $\Omega \quad \Rightarrow \quad \mathrm{RLCT}_\Omega(f;\varphi) = \mathrm{RLCT}_\Omega(g;\varphi)$.

Proposition. $K(\omega) \sim \sum_{i=1}^k (p_i(\omega) - q_i)^2$ over Ω

(Proof: Full rank of Hessian of $K(p) = \sum q_i \log \frac{q_i}{p_i}, \sum q_i = \sum p_i = 1$.)

Remark. Similar sum of squares in several statistical scenarios. e.g. Multivariate Gaussian

$$K(\omega) = \frac{1}{2} \log \det \Sigma(\omega) - \frac{1}{2n} \sum_{i=1}^n (x_i - \mu(\omega))^T \Sigma^{-1} (x_i - \mu(\omega))$$
$$\sim \sum_{i=1}^k (\mu_i(\omega) - \hat{x}_i)^2 + \sum_{i=1}^k \sum_{j=1}^k (\Sigma_{ij}(\omega) - \hat{\Sigma}_{ij})^2$$

Proposition. If $\langle f_1, \ldots, f_r \rangle = \langle g_1, \ldots, g_s \rangle$ as ideals, then $\mathrm{RLCT}_\Omega(\sum f_i^2;\varphi) = \mathrm{RLCT}_\Omega(\sum g_i^2;\varphi)$

Definition. Given ideal $I = \langle f_i \rangle$, define $\mathrm{RLCT}_\Omega(I;\varphi) = \mathrm{RLCT}_\Omega(\sum f_i^2;\varphi)$.

Corollary. $\mathrm{RLCT}_\Omega(K;\varphi) = \mathrm{RLCT}_\Omega(\langle p_i(\omega) - q_i \rangle;\varphi)$.

5.2 Nondegeneracy of ideals

Proposition. If $\langle f_1, \ldots, f_r \rangle = \langle g_1, \ldots, g_s \rangle$ as ideals, then $\sum f_i^2$ is ndg iff $\sum g_i^2$ is ndg.

Definition. An ideal $I = \langle f_i \rangle$ is ndg if $\sum f_i^2$ is ndg.

Proposition. Monomial ideals are ndg.

Corollary. The RLCT of monomial ideals can be computed from their Newton polyhedra.

5.3 Higher Order Terms

Let $K(p)$ real analytic at origin, $p(\omega)$ real analytic map, Ω small nbd of origin,

$$Z(n) = \int_\Omega e^{-nK(p(\omega))} \varphi(\omega) d\omega.$$

We have seen before that the full asymptotics of $Z(n)$ can be computed by resolving $K(p(\omega))$. Under certain conditions on K, the RLCT can be computed by resolving the ideal $\langle p_i(\omega) \rangle$.

Proposition. If $K(0) = 0$, $\partial K(0) = 0$ and $\partial^2 K(0)$ has full rank, then,

$$K(p(\omega)) \sim \sum p_i(\omega)^2$$

$$K(p(\omega)) \text{ ndg at } \omega_0 \text{ iff } \sum p_i(\omega)^2 \text{ ndg at } \omega_0$$

for all real analytic maps p and $\omega_0 \in p^{-1}(0)$.

Corollary. Assume K as before. The full asymptotics of $Z(n)$ can be computed by resolving $\langle p_i(\omega) \rangle$.