Relations between Principal Minors of a 4×4 Matrix

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[About the problem I am presenting today: it arose out of some email correspondences between Bernd Sturmfels and Harald Helfgott (University of Bristol, UK). Helfgott wanted to find polynomial relations satisfied by the principal minors of a 4×4 matrix. We played around with the problem, and even used the program `Singular`, but with no success. Eventually, Helfgott discovered some papers by E. J. Nanson and Thomas Muir from the 1890s with the solution. These classical papers seem to have been forgotten, but they use some really cool tricks to solve the problem. My goal is to update them to modern language, and to apply the tricks to similar problems.]

[The outline of the talk will be as follows: first, I will introduce some notation. Next, I will give the background of the problem. Finally, I will explain the tricks used in the classical papers to solve the problem.]

[Feel free to stop me for questions at any time. Let me start with some definitions.]

1 Definitions

Let $A = (a_{ij})$ be a $n \times n$ matrix, $A \in \mathbb{C}^{n^2}$.

principal minor: a minor with rows, columns indexed by same subset $I \subset [n] = \{1, ..., n\}$.

A_I: minor of matrix A indexed by I. $A_\emptyset := 1$.

$$\begin{pmatrix} \times & \times & \cdot & \times \\ \times & \times & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \times \end{pmatrix} \qquad A_{124} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

$A_* \in \mathbb{C}^{2^n}$: vector of PMs of A.

principal minor map $\phi : \mathbb{C}^{n^2} \to \mathbb{C}^{2^n}, A \mapsto A_*$.

$P_n \in \mathbb{C}[A_*]$: prime ideal of all poly. relns among the PMs. $P_n = \mathcal{I}(\text{Im } \phi)$

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 QN: why is it a prime ideal?
ANS: any variety defined by a rational parametrization
     is irreducible, so its ideal is prime.
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2 Background

Principal Minor Assignment Problem (PMAP)
Given $A_* \in \mathbb{C}^{2^n}$, determine whether $A_* \in \text{Im } \phi$.

[Being algebraic geometers, we are interested in solving PMAP algebraically.]

Recall constructible set
$Z = X_0 - X_1 + \cdots + (-1)^k X_k$
where $X_0 \supset X_1 \supset \cdots \supset X_k$ are varieties.

[Chevalley's theorem tells us that the image of a constructible set under a regular map is also constructible.]

[Contructible Set (alternative equivalent definition):
finite disjoint union of quasiprojective varieties (locally closed subsets).]

[Chevalley's Thm: Let $X \subset \mathbb{P}^m$ be a quasi-projective variety, $f : X \to \mathbb{P}^n$ a regular map, and $U \subset X$ any constructible set. Then, $f(U)$ is a constructible subset of $\mathbb{P}^n$.]

Chevalley's Thm $\Longrightarrow$ $\text{Im } \phi = X_0 - X_1 + \cdots + (-1)^k X_k$
where $X_0 \supset X_1 \supset \cdots \supset X_k$ are varieties.

Principal Minor Generator Problem (Open)
Determine a finite set of generators for the each ideal $\mathcal{I}(X_i)$.

$\longrightarrow$ an algebraic solution to PMAP.

Griffin, Tsatsomeros (2006): algorithmic solution to PMAP under "genericity" condition.

[Kent G., Michael T., Washington State University]

If A_* satisfies "genericity" conditions, then algorithm either:
(i) outputs a solution matrix, or
(ii) determines that none exist.

3 Today's problem

[I've mentioned a lot of interesting problems in my introduction of the background, but don't let that distract you from the main problem that I want to solve today.]

Today's problem
Find relations in $P_4 = \mathcal{I}(X_1)$, where $X_1 = \overline{\text{Im } \phi}$

Why $n = 4$?

- No. of matrix entries: n^2.
- A_* invariant under conjugation of A by diagonal matrices: $(n-1)$ degrees of freedom.
  [Show why this is true for 4x4 determinant. If the diagonal matrix is scaled, its action on A does not change.]
- $\therefore \dim \text{Im } \phi \leq n^2 - n + 1$.
- Dim. of space of PMs (ignore $A_\emptyset$): $2^n - 1$.
- For $n < 4, n^2 - n + 1 \geq 2^n - 1$.
  [In fact, they are equal. This does not prove that there are no polynomial relations, but one can easily prove that for $n = 1, 2, 3$, all vectors of length $2^n - 1$ are realizable as the principal minors of some matrix.]
- For $n = 4, n^2 - n + 1 = 13, 2^n - 1 = 15$. First non-trivial case.

Holtz, Sturmfels (2007): algebraic solution to symmetric 4x4 PMAP.

[Now both with UC Berkeley. Explain "symmetric".]

[Showed that for symmetric matrices, the image of the principal minor map is a Zariski closed. 20 relations derived from hyperdeterminant. A vector A_* is realizable as principal minors iff it satisfies these 20 relations. (Showed the relations are in the same orbit of a group action that leaves P_n invariant.)]

[Going back to our problem for the general 4x4 matrix, how can we find relations satisfied by the principal minors? One way is to use

Groebner bases

methods from elimination theory that allow us to compute the generators for this ideal. Dustin and I used

`Singular`

to do this, but the math server ran out of memory before the computation was completed.]

[We need a new approach, using so-called devertebrated minors.]

4 Devertebrated Minors

[This is the approach discovered by

E. J. Nanson, Thomas Muir, 1897-98

What are devertebrated minors?]

Given a matrix A, replace diagonal elements by zeroes. [Call this new matrix...]

$$B = \begin{pmatrix} \cdot & a_{12} & a_{13} & a_{14} \\ a_{21} & \cdot & a_{23} & a_{24} \\ a_{31} & a_{32} & \cdot & a_{34} \\ a_{41} & a_{42} & a_{43} & \cdot \end{pmatrix}$$

The principal minors $B_I, |I| > 1$ [of new matrix B] are the **devertebrated minors** of A.

$$B_{23} = \begin{pmatrix} \cdot & a_{23} \\ a_{32} & \cdot \end{pmatrix} = -a_{23}a_{32}$$

[achieved by taking the principal minor A_{23}
and replacing the diagonals with zero.]

[There is a nice relationship between the principal minors and devertebrated minors.]

Proposition 1. *(Cayley) The PMs of A are poly. fns of the DMs and diag. elements.*

$$A_I = \sum_{S \sqcup T = I} B_S \prod_{t \in T} A_t$$

[Here, S and T can be empty. Conversely, ...]

Proposition 2. *(Muir) The DMs (and diag. elements) of A are poly. fns of the PMs.*

$$B_I = \sum_{S \sqcup T = I} (-1)^{|T|} A_S \prod_{t \in T} A_t$$

[(Proof also by inclusion-exclusion principle). Hence, any relation between the devertebrated minors can be translated into a relation between the principal minors. We will now focus on finding relations between the devertebrated minors.]

[The next trick uses trigonometry.]

5 Using Trigonometry

[We start with the following useful...]

Lemma 3. *If* $\frac{x}{y} = e^{2i\theta}$*, then* $x + y = 2\sqrt{xy}\cos\theta$*.*

Proof. Use $2\cos^2\theta - 1 = \cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta})$.

□

Notation: let $c_{i_1 i_2 \ldots i_k}$ denote the [cyclic] product $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}$.

[Define]

$$\frac{c_{234}}{c_{243}} = e^{2i\alpha}, \quad \frac{c_{134}}{c_{143}} = e^{2i\beta}, \quad \frac{c_{124}}{c_{142}} = e^{2i\gamma}$$

[Then,]

$$\begin{aligned}
B_{234} &= c_{234} + c_{243} = 2\sqrt{-B_{23}B_{34}B_{42}}\cos\alpha && (1)\\
B_{134} &= c_{134} + c_{143} = 2\sqrt{-B_{13}B_{34}B_{41}}\cos\beta && (2)\\
B_{124} &= c_{124} + c_{142} = 2\sqrt{-B_{12}B_{24}B_{41}}\cos\gamma && (3)\\
B_{123} &= c_{123} + c_{132} = 2\sqrt{-B_{12}B_{23}B_{31}}\cos(\alpha + \beta + \gamma) && (4)
\end{aligned}$$

[The last equation follows because]

$$\frac{c_{123}}{c_{132}} = e^{2i(\alpha+\beta+\gamma)}$$

Also,

$$\begin{aligned}
B_{1234} - B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23} &= -(c_{1243} + c_{1342})\\
&\quad -(c_{1234} + c_{1432})\\
&\quad -(c_{1423} + c_{1324})\\
&= -2\sqrt{B_{12}B_{24}B_{43}B_{31}}\cos(\beta + \gamma)\\
&\quad -2\sqrt{B_{12}B_{23}B_{34}B_{41}}\cos(\alpha + \gamma)\\
&\quad -2\sqrt{B_{14}B_{42}B_{23}B_{31}}\cos(\alpha + \beta)
\end{aligned}$$

since

$$\frac{c_{1243}}{c_{1342}} = \cdot\frac{c_{134}}{c_{143}} \cdot \frac{c_{124}}{c_{142}} = e^{2i(\beta+\gamma)}, \quad \text{and so on.}$$

$$
\begin{aligned}
x &= -\sqrt{B_{12}B_{24}B_{43}B_{31}}\cos(\beta+\gamma)\\
y &= -\sqrt{B_{12}B_{23}B_{34}B_{41}}\cos(\alpha+\gamma)\\
z &= -\sqrt{B_{14}B_{42}B_{23}B_{31}}\cos(\alpha+\beta)
\end{aligned}
$$

$$
\begin{pmatrix}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
2 & 2 & 2 & B_{12}B_{34}+B_{13}B_{24}+B_{14}B_{23}-B_{1234}
\end{pmatrix}
\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \mathbf{0}
$$

Expand (4) with

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos(\beta+\gamma)+\cos\beta\cos(\alpha+\gamma)+\cos\gamma\cos(\alpha+\beta)-2\cos\alpha\cos\beta\cos\gamma,$$

$\Longrightarrow$

$$B_{234}B_{14}x + B_{134}B_{24}y + B_{124}B_{34}z + (B_{123}B_{14}B_{24}B_{34} - \frac{1}{2}B_{234}B_{134}B_{124}) = 0 \qquad (5)$$

[In a similar way, we can find expansions for $\cos\alpha$, etc. which gives us three more equations in terms of x, y, z. We get the following system of equations.]

$$
\begin{pmatrix}
B_{123}B_{14} & B_{124}B_{13} & B_{134}B_{12} & B_{234}B_{12}B_{13}B_{14} - \frac{1}{2}B_{134}B_{124}B_{123}\\
B_{124}B_{23} & B_{123}B_{24} & B_{234}B_{21} & B_{134}B_{21}B_{23}B_{24} - \frac{1}{2}B_{234}B_{124}B_{123}\\
B_{134}B_{32} & B_{234}B_{31} & B_{123}B_{34} & B_{124}B_{31}B_{32}B_{34} - \frac{1}{2}B_{234}B_{134}B_{123}\\
B_{234}B_{41} & B_{134}B_{42} & B_{124}B_{43} & B_{123}B_{41}B_{42}B_{43} - \frac{1}{2}B_{234}B_{134}B_{124}\\
2 & 2 & 2 & B_{12}B_{34}+B_{13}B_{24}+B_{14}B_{23}-B_{1234}
\end{pmatrix}
\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \mathbf{0}
$$

[Deleting any row of the above matrix gives a 4x4 matrix with a non-zero null space. Thus, the determinant of each new 4x4 matrix is zero.]

Every 4×4 maximal minor of the above matrix vanishes.

6 Computations

[Translating the relations between the DMs into those between the PMs gives]

4 polynomials of deg 12 having 5234 terms, (weighted deg. 19)
1 polynomial of deg 16 having 19012 terms, (weighted deg. 24)

In B_I, 4 polynomials of deg 8 having 47 terms, (weighted deg. 19)
1 polynomial of deg 10 having 19 terms, (weighted deg. 24)
Variety is of dimension 13.

7 Future Direction and Conclusion

1. Find out if the above relations generate P_4
2. Find generators for $P_n, \quad n > 4$.
3. Solve Principal Minor Generators Problem.

[To summarize: two tricks, using devertebrated minors and trigonometry, were used to solve this sub-problem. These methods may be useful for other determinantal-type problems. Thank you. Any questions?]

8 Acknowledgements

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9 Reduced PMAP

Let A_* be a vector of length 2^n, and let B_* be the vector of length $2^n - n$ obtained from A_* by the relations in Proposition 7.2. Then, A_* is realizable as the principal minors of some $n \times n$ matrix iff B_* is realizable as the principal minors of some devertebrated $n \times n$ matrix.

References

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