Relations among Principal Minors of a Matrix

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Definitions

Let $A = (a_{ij}) \in \mathbb{C}^{n^2}$ be a complex $n \times n$ matrix.

- Principal minor: a minor with rows and columns indexed by same subset $\sigma \subset [n] := \{1, \ldots, n\}$.

  $A_\sigma \in \mathbb{C}$: principal minor of A indexed by σ, with $A_\emptyset = 1$.

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- **Related Problem**: Find a finite generating set for P^a_n.

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- Principal Minor Assignment Problem (PMAP)
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  - Gröbner basis methods infeasible.
- Principal Minors of Symmetric Matrix
  - Holtz & Sturmfels [3], 2007: studied relations among principal minors of a *symmetric* matrix.
  - Found links to hyperdeterminant.

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- For $n = 4$, this vector of principal minors is not realizable:

$$A_{123} = A_{124} = A_{134} = A_{234} = 1, A_1 = A_2 = \ldots = A_{1234} = 0$$

Hence,

$$P_4 \neq \{0\}.$$

Some relations were found by Nanson & Muir in 1897.
But they do not generate P_4.

Cycles and Cycle-sums

Let $X = (x_{ij})$ be an $n \times n$ matrix of indeterminates.

- Given a permutation π of $\{i_1, \ldots, i_k\} \subset [n]$, define

  $$c_\pi = x_{i_1\pi(i_1)} x_{i_2\pi(i_2)} \cdots x_{i_k\pi(i_k)}.$$

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- Given a partition λ of $\{i_1, \ldots, i_k\} \subset [n]$, define

$$C_\lambda = \sum_{\pi \in S_\lambda} x_{i_1\pi(i_1)} x_{i_2\pi(i_2)} \cdots x_{i_k\pi(i_k)}.$$

  where S_λ consists of permutations whose cycles give the partition λ.
  We call C_λ a cycle-sum if λ has only one part.
  There are 2^n cycle-sums, one for each subset of $[n]$ (define $C_\emptyset = 1$).

Usefulness of Cycles and Cycle-sums

1. Closure of $\operatorname{Im}\phi^a$

Theorem 1. $\operatorname{Im}\phi^a$ is closed.

This means that a vector of length $2^n - 1$ is realizable as the principal minors of some $n \times n$ matrix iff it satisfies the relations in P_n^a.

Hence, PMAP is solved if we find finite generating sets for each P_n^a.

Usefulness of Cycles and Cycle-sums

2. Finding relations among principal minors

Proposition 2. The principal minors and cycle-sums satisfy

$$A_\lambda = \sum_{\lambda=\lambda_1 \sqcup \ldots \sqcup \lambda_k} (-1)^{k+d} C_{\lambda_1} \cdots C_{\lambda_k}$$

$$C_\lambda = \sum_{\lambda=\lambda_1 \sqcup \ldots \sqcup \lambda_k} (-1)^{k+d} (k-1)! A_{\lambda_1} \cdots A_{\lambda_k}$$

where $\lambda \subset [n]$, $|\lambda| = d$ and the $\lambda_1 \sqcup \ldots \sqcup \lambda_k$ are partitions of λ.

Corollary 3. Let $\psi : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$, $A_* \mapsto C_*$. Then, $\boldsymbol{u} \in \mathbb{C}^{2^n}$ is realizable as principal minors iff the $\psi(\boldsymbol{u})$ is realizable as cycle-sums.

New approach: Find relations among cycle-sums.

Relations among Cycle-sums

Let $x = c_{1243} + c_{1342}$, $y = c_{1234} + c_{1432}$, $z = c_{1423} + c_{1324}$.

Then,

$C_{1234} = x + y + z$
$2C_{234}C_{12}C_{13}C_{14} + C_{134}C_{124}C_{123} = C_{123}C_{14}x + C_{124}C_{13}y + C_{134}C_{12}z$.

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$$\begin{pmatrix} C_{123}C_{14} & C_{124}C_{13} & C_{134}C_{12} & 2C_{234}C_{12}C_{13}C_{14} + C_{134}C_{124}C_{123} \\ C_{124}C_{23} & C_{123}C_{24} & C_{234}C_{21} & 2C_{134}C_{21}C_{23}C_{24} + C_{234}C_{124}C_{123} \\ C_{134}C_{32} & C_{234}C_{31} & C_{123}C_{34} & 2C_{124}C_{31}C_{32}C_{34} + C_{234}C_{134}C_{123} \\ C_{234}C_{41} & C_{134}C_{42} & C_{124}C_{43} & 2C_{123}C_{41}C_{42}C_{43} + C_{234}C_{134}C_{124} \\ 1 & 1 & 1 & C_{1234} \end{pmatrix} \begin{pmatrix} -x \\ -y \\ -z \\ 1 \end{pmatrix} = \mathbf{0}$$

Since the matrix equation has a non-trivial solution,
all the 4×4 minors of the matrix must vanish.
Let I be the ideal generated by these maximal minors.

Relations among Cycle-sums

Define $g = \begin{vmatrix} C_{123}C_{14} & C_{124}C_{13} & C_{134}C_{12} \\ C_{124}C_{23} & C_{123}C_{24} & C_{234}C_{21} \\ C_{134}C_{32} & C_{234}C_{31} & C_{123}C_{34} \end{vmatrix}$

Theorem 4. The ideal P_4^a of relations among the principal minors of a 4×4 matrix is the ideal quotient $I : g$.

Remark: P_4^a is minimally generated by 65 polynomials of deg 12.

Group Action on Principal Minors

- Isomorphism between $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and vector space generated by principal minors. e.g.

$$\begin{aligned} A_{123} &\leftrightarrow \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \\ aA_{23} + bA_{123} &\leftrightarrow \begin{pmatrix}a\\b\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \end{aligned}$$

- Action of $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ induces action on principal minors that is invariant on P_4.
- **Idea:** Use group action to discover all the relations in P_4 from P_4^a.

Hyperdeterminantal Relations

Define
$$
\begin{aligned}
F \quad = \quad & A_\emptyset + A_1 x + A_2 y + A_3 z + A_4 w \\
& + A_{12}xy + A_{13}xz + A_{14}xw + A_{23}yz + A_{24}yw + A_{34}zw \\
& + A_{123}xyz + A_{124}xyw + A_{134}xzw + A_{234}yzw + A_{1234}xyzw.
\end{aligned}
$$

Definition 5. The hyperdeterminant D_{2222} is the unique irreducible polynomial (up to sign) of content one in the unknowns A_* which vanishes whenever $F = \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = \partial F/\partial w = 0$ has a solution (x_0, y_0, z_0, w_0) in $\mathbb{C}^4$.

The irreducible components of the singular locus of the hypersurface $D_{2222} = 0$ were classified by Weyman and Zelevinsky in 1996.

Conjecture 6. $\operatorname{Im}\phi$ is the irreducible component $\nabla_{\mathrm{node}}(\emptyset)$.

Conclusion

- Usefulness of cycles and cycle-sums.
  - Closure of $\mathrm{Im}\,\phi^a$.
  - Minimal generators of P_4^a.
- Group action on principal minors to find P_4.
- Relationship with hyperdeterminants.

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Open Questions:

- Is the hyperdeterminantal conjecture true?
- What is a finite generating set for P_4?
- Can we use cycle-sums to find relations in $P_n, n > 4$?

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