Math 1B Section 101
28 Oct 2009 Quiz #9 (15 min)

Name: ______________________________ Score: /10

Explain your answers to the following questions *clearly*.

Let $T(x)$ be the Taylor series of $\cos^2 x$ around $x = 0$.
Let $T_n(x)$ be the n-th degree Taylor polynomial.

1. (3 points) Find $T_6(x)$.

2. (2 points) Find $T(x)$. Show clearly your formula for a general term in the series.

[Please turn over.]

3. (2 points) What is the radius of convergence for $T(x)$? Explain.

4. (3 points) Approximate $\cos^2(0.3)$ using $T_4(x)$, giving your answer as a decimal. Estimate the error. (Reference: $3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, 3^6 = 729$)

[End of Quiz]

Quiz #9 Answers

① Find $T_6(x)$.

<u>Method 1</u>: Write $\cos^2 x = \dfrac{\cos(2x)+1}{2}$

$$\cos^2 x = \frac{\cos 2x + 1}{2} = \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots\right) + \frac{1}{2}$$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots$$

<u>Method 2</u>: multiplying power series.

$$\cos^2 x = (\cos x)(\cos x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$
$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

	1	$-\frac{x^2}{2}$	$+\frac{x^4}{24}$	$-\frac{x^6}{720}$	$\dots$
1	1	$-\frac{1}{2}$	$\frac{1}{24}$	$-\frac{1}{720}$	
$-\frac{x^2}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{48}$	$\dots$	
$+\frac{x^4}{24}$	$\frac{1}{24}$	$-\frac{1}{48}$	$\dots$		
$-\frac{x^6}{720}$	$-\frac{1}{720}$	$\dots$			
$\vdots$					

1

$(-1)x^2$

$(\frac{1}{3})x^4$

$(-\frac{2}{45})x^6$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots$$

Method 3: Taylor formula.

$f(x) = \cos^2 x$ $\qquad f(0) = 1$

$f^{(1)}(x) = -2\cos x \sin x$ $\qquad f^{(1)}(0) = 0$

$f^{(2)}(x) = -[2(-\sin x)\sin x + 2\cos x(\cos x)]$

$= 2(\sin^2 x - \cos^2 x)$ $\qquad f^{(2)}(0) = -2$

$f^{(3)}(x) = 2[2\cos x \sin x + 2\sin x \cos x]$

$= 8\cos x \sin x$ $\qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = -8(\sin^2 x - \cos^2 x)$ $\qquad f^{(4)}(0) = 8$

$f^{(5)}(x) = -32\cos x \sin x$ $\qquad f^{(5)}(0) = 0$

$f^{(6)}(x) = 32(\sin^2 x - \cos^2 x)$ $\qquad f^{(6)}(0) = -32$

$$\therefore f(x) = 1 - \frac{2}{2!}x^2 + \frac{8}{4!}x^4 - \frac{32}{6!}x^6 + \cdots$$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots$$

② Find $T(x)$.

Method 1: $\cos^2 x = \dfrac{\cos 2x + 1}{2}$

$$\cos^2 x = \frac{\cos 2x + 1}{2} = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!}$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n \frac{2^{2n-1}}{(2n)!}x^{2n}$$

Method 2: Multiplication of power series.

$$\cos^2 x = (\cos x)(\cos x)$$
$$= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right)$$

The coefficient of x^{2n} in the expansion is

$$(-1)^n \left[\frac{1}{0!}\cdot\frac{1}{(2n)!} + \frac{1}{2!}\cdot\frac{1}{(2n-2)!} + \cdots + \frac{1}{(2n-2)!}\cdot\frac{1}{2!} + \frac{1}{(2n)!}\cdot\frac{1}{0!}\right]$$

$$= (-1)^n \sum_{k=0}^{n} \frac{1}{(2k)!}\cdot\frac{1}{(2n-2k)!}$$

Thus, $\cos^2 x = \sum_{n=0}^{\infty} (-1)^n \left[\sum_{k=0}^{n} \frac{1}{(2k)!}\cdot\frac{1}{(2n-2k)!}\right] x^{2n}$

Bonus: $\sum_{k=0}^{n} \frac{1}{(2k)!}\cdot\frac{1}{(2n-2k)!} = \sum_{k=0}^{n} \frac{1}{(2n)!}\binom{2n}{2k} = \frac{2^{2n-1}}{(2n)!}$

$$(1+x)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i$$

$$(1-x)^{2n} = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} x^i$$

$$(1+x)^{2n} + (1-x)^{2n} = 2\sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n}{i} x^i = 2\sum_{k=0}^{n} \binom{2n}{2k} x^{2k}$$

Substitute $x=1$. Then $2^{2n} = 2\sum_{k=0}^{n}\binom{2n}{2k}$.

Thus, $\sum_{k=0}^{n}\binom{2n}{2k} = 2^{2n-1}$

Method 3: Taylor formula.

$f(x) = \cos^2 x$

$f^{(1)}(x) = (-2)(\cos x \sin x)$

$f^{(2)}(x) = (-2)(\cos^2 x - \sin^2 x)$

$f^{(3)}(x) = (8)(\cos x \sin x)$

$f^{(4)}(x) = (8)(\cos^2 x - \sin^2 x)$

$f^{(5)}(x) = (-32)(\cos x \sin x)$

$f^{(6)}(x) = (-32)(\cos^2 x - \sin^2 x)$.

By induction,

$f^{(2n-1)}(x) = (-1)^n 2^{2n-1}(\cos x \sin x)$ $\qquad f^{(2n-1)}(0) = 0$

$f^{(2n)}(x) = (-1)^n 2^{2n-1}(\cos^2 x - \sin^2 x)$ $\qquad f^{(2n)}(0) = (-1)^n 2^{2n-1}$

Thus, $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{2n}$

③ Radius of convergence for $T(x)$.

Method 1: There is a theorem which says that if $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are power series convergent for all $|x| < R$, then their product is a power series that converges for all $|x| < R$. Since $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converges for all x,

the product $(\cos x)(\cos x) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right)^2$
also converges for all x. Hence, the radius of convergence is ∞.

<u>Method 1:</u> $\cos^2 x = \frac{\cos 2x + 1}{2}$

The power series $\cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}$ converges for all values of $2x$, so the power series for $\cos^2 x$ converges for all x. Thus, the radius of convergence is ∞.

<u>Method 3:</u> Ratio Test.

Let $a_n = (-1)^n \frac{2^{2n-1}}{(2n)!} x^{2n}$

Then, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^2 |x|^2}{(2n+2)(2n+1)} = 0 < 1$

Hence, by the Ratio Test, the series $\sum_{n=0}^{\infty} a_n$ converges for all x. Thus, the radius of convergence is ∞.

(4) Approximate $\cos^2(0.3)$ using $T_4(x)$.

$$T_4(0.3) = 1 - (0.3)^2 + \frac{1}{3}(0.3)^4$$

$$= 1 - 0.09 + \frac{1}{3}(0.0081)$$

$$= 0.91 + 0.0027$$

$$= 0.9127.$$

Using the Alternating Series Estimate Theorem, we first note that $\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n-1}}{(2n)!}(0.3)^{2n}$ is an alternating series because for $b_n = \frac{2^{2n-1}}{(2n)!}(0.3)^{2n}$,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2}\frac{[(2)(0.3)]^{2n}}{(2n)!} = 0$$

(since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all real x)

$$b_{n+1} = b_n \cdot \frac{(2^2)(0.3)^2}{(2n+1)(2n+2)} = b_n \frac{0.36}{(2n+1)(2n+2)} < b_n$$

so b_n is a decreasing sequence.

Thus,

$$|\text{Error}| < b_3 = \frac{2}{45}(0.3)^6 = 0.0000324.$$

(4) Method 2: Using Taylor's Inequality
(Not recommended for this question)

We consider the range $|x| \leq 0.3$

In this range,

$$|R_5(x)| \leq \frac{M}{5!}|x|^5$$

where $M = \max\limits_{|x|\leq 0.3} |f^{(5)}(x)| = \max\limits_{|x|\leq 0.3} |(-32)\cos x \sin x|$

$$= \max_{|x|\leq 0.3} |16 \sin 2x|$$

$$= 16 \sin 0.6.$$

Thus, $|R_5(x)| \leq \frac{16\sin(0.6)}{5!}|x|^5$

$$\leq \frac{16 \sin(0.6)}{5!}(0.3)^5 \simeq 0.0001829$$

Quiz Statistics

Scores	0	1	2	3	4	5	6	7	8	9	10
	3	0	2	3	8	2	4	5	1	2	0

Average 4.67

Grading Scheme

Q1)
(1 pt) differentiating and substituting $x = 0$.
(1 pt) correct general formula for Taylor series.
(1 pt) correct formula for $T_6(x)$.

Q2)
(1 pt) method: multiplying, Taylor formula or half angle formula.
(1 pt) correct general formula.

Q3)
(1 pt) radius is infinity.
(1 pt) correct reason: ratio test or product of convergent series.

Q4) (1 pt) substituting $x = 0.3$ in $T_4(x)$.
(1 pt) method for computing error: Taylor's inequality or Alt. Series Est. Thm.
(1 pt) correct error.

Good Observations

1. The student writes down the general idea for solving the problem, even though he/she could not carry it out. This allows the grader to give some points.

2. Because of the lack of time, the student writes down the big ideas and equations rather than full sentences. After finishing the quiz, the student goes through to fill in the details.

3. For $f(x) = \cos x$, the student realizes that $f'(x) = -2\sin x \cos x = -\sin(2x)$. After that, the derivatives $f''(x), f'''(x), \ldots$ were a lot easier to find and generalize.

Common Mistakes

Q1 Wrong differentiation (Arithmetic mistakes).

Q1 The student does not simplify the derivatives, so he/she cannot see the pattern.

Q1 The square of a power series $\sum c_n x^n$ is not $\sum c_n^2 x^{2n}$.

Q2 Not writing the answer in a form such that the coefficient of a general term is obvious.

Q2 The student finds a pattern in the coefficients and says the general term follows "by induction" even though the proof of this general formula is not clear.

Usually, when we say "by induction," it means that we found a pattern that *we can prove* to hold in general. We don't write down the proof because it is tedious to do it in detail, but it is clear from the pattern.

e.g. $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = \sin x$, etc. so by induction, $f^{(2n)}(x) = (-1)^n \sin x, f^{(2n+1)}(x) = (-1)^n \cos x$.

Q3 It does not make sense to talk about the radius of convergence *of a function*, e.g. $\cos x$. What we mean is the radius of convergence for a *power series* representing the function.

Q3 Forgetting to state you used the ratio test to find radius of convergence.

Q4 $T_4(x)$ is not the sum of the first 4 non-zero terms, but the sum up to the degree-4 term.

Q4 Not simplifying the approximation and giving the answer as a decimal.

Q4 It is better to use the Alternating Series Estimate Theorem rather than Taylor's inequality for estimating the error, because we are finding the error at one point $x = 0.3$ rather than the error in a range, say $-0.2 \leq x \leq 0.2$.

Q4 To approximate $\cos^2(0.3)$, do not expand $\cos^2 x$ around $x = 0.3$ because we do not know $f(0.3), f'(0.3), f''(0.3)$, etc. for $f(x) = \cos^2 x$.

Q4 The student finds $T_4(x)$ from scratch, i.e. differentiating and using Taylor formula, etc.! You can just get it from $T_6(x)$.

Q4 Finding the approximation but forgetting to estimate the error (perhaps the student ran out of time).

Q4 For the Alternating Series Estimate Theorem, the next term b_{n+1} is $(2/45)(0.3)^6$ and not $f^{(5)}(0.3)$. The student is mixing the Taylor's inequality with this theorem.

Q1-4 Not striking out conflicting answers. Points will be deducted even if one of your answers was completely correct! On the other hand, if you do not know the correct answer, you should write down everything you know or have tried, so at least perhaps some partial credit can be given.