Math 1B Section 101
21 Oct 2009 Quiz #8 (15 min)

Name: ____ Score: /10

1. (6 points) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty}(-1)^{3n+1}\frac{(3x+1)^{3n+1}}{3n+1}.$$

Explain your answer clearly.

Ans:
Let $a_n = (-1)^{3n+1}(3x+1)^{3n+1}/(3n+1)$.

We apply the Ratio Test to find the radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3n+1}{3n+4}\ |3x+1|^3 = |3x+1|^3$$

where $\lim_{n\to\infty}(3n+1)/(3n+4) = 1$ because the limit of a ratio of two polynomials of the same degree is the ratio of their leading coefficients.

By the Ratio Test, $\sum_{n=1}^{\infty} a_n$ converges if $|3x+1|^3 < 1$ and diverges if $|3x+1|^3 > 1$.

$$|3x+1|^3 < 1 \quad\Rightarrow\quad |3x+1| < 1 \quad\Rightarrow\quad |x+1/3| < 1/3$$

Thus, the center of the power series is $x = -1/3$ and its radius of convergence is $1/3$.

To find the interval of convergence, we test the end-points. When $x = -2/3$,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{3n+1}$$

which diverges by a Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} 1/n$:

$$\lim_{n\to\infty}\frac{1/(3n+1)}{1/n} = \lim_{n\to\infty}\frac{n}{3n+1} = 1/3 > 0.$$

When $x = 0$,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3n+1}$$

which converges by the Alternating Series Test, since $1/(3n+1)$ is positive, tends to 0 as $n\to\infty$ and is a decreasing function. Thus, the interval of convergence is $(-2/3, 0]$.

[Please turn over.]

2. (4 points) Evaluate the indefinite integral as a power series, showing clearly your formula for a general term in the series. What is the radius of convergence?

$$\int \frac{1}{(1+2x)(1-2x)}\,dx$$

Ans:
Using the power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

we get the expansion

$$\begin{aligned}\frac{1}{(1-2x)(1+2x)} = \frac{1}{1-4x^2} &= 1 + 4x^2 + (4x^2)^2 + (4x^2)^3 + \cdots \\ &= 1 + 4x^2 + 4^2x^4 + 4^3x^6 + \cdots .\end{aligned}$$

Thus, applying term-by-term integration, the indefinite integral equals

$$\begin{aligned}\int 1 + 4x^2 + 4^2x^4 + 4^3x^6 + \cdots\,dx &= C + x + \frac{4x^3}{3} + \frac{4^2x^5}{5} + \frac{4^3x^7}{7} + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{4^n}{2n+1} x^{2n+1}\end{aligned}$$

where C is a constant.

As for the radius of convergence, the series for $1/(1-x)$ converges for $|x| < 1$ and diverges for $|x| > 1$. Thus, the series for $1/(1-4x^2)$ converges for $|4x^2| < 1 \Rightarrow |x| < 1/2$ and diverges for $|x| > 1/2$. Since the radius of convergence does not change under term-by-term integration, the radius for the series of the indefinite integral is $1/2$.

[End of Quiz]

Quiz Statistics

Scores	0	1	2	3	4	5	6	7	8	9	10
	1	0	0	1	5	7	8	5	2	0	1

Average 5.57

Grading Scheme

Q1)
(1 pt) idea of using ratio test.
(1 pt) correct limit $|3x+1|^3$ and stating you used ratio test.
(1 pt) proving $\lim(3n+1)/(3n+4) = 1$.
(1 pt) correctly showing radius is $1/3$.
(1 pt) correctly showing series divergent at $x = -2/3$, say using LCT.
(1 pt) correctly showing series convergent at $x = 0$, say using AST.

Q2)
(1 pt) idea of using expansion $1/(1-x) = \sum x^n$.
(1 pt) idea of using partial fractions or $1/(1-4x^2)$.
(1 pt) idea of integrating term by term.
(-1 pt) arithmetic mistakes in any of the above three steps, including not writing "+C".
(1 pt) correctly showing radius is $1/2$, using ratio test or the fact that radius is unchanged after integration of power series.

Common Mistakes

Q1 Not realizing that $(-1)^{6n+2} = 1$.
Q1 $|3x+1| < 1$ does not imply $|3x| < 0$.
Q1 The $(n+1)$-th term is $(3x+1)^{3n+4}/(3n+4)$, not $(3x+1)^{3n+2}/(3n+2)$.
Q1 Not explaining why $\lim(3n+1)/(3n+4) = 1$.
Q1 $|x+1/3| < 1/3$ implies center of power series is $x = -1/3$, not $x = 1/3$.
Q1 $|3x+1| < 1$ implies radius is $1/3$, not 1.
Q1 Not stating you used Ratio Test.
Q1 Stating but not explaining convergence, divergence of endpoints.
Q1 For limit of ratio of polynomials $\lim_{x\to\infty} p(x)/q(x)$ with same degree, the limit is not 1 but the ratio of the leading coefficients.
Q2 Forgetting "+C" or saying "C=0" for this indefinite integral.
Q2 Did not state that the radius of convergence is unchanged after integrating power series.
Q2 Integral of $(2x)^n$ is not $(2x)^{n+1}/(n+1)$ (Most common mistake!!)
Q2 Not writing the general formula for the power series in the answer.
Q2 Forgetting to integrate power series.
Q2 The sum $\sum c_n x^n$ of two power series $\sum a_n x^n$, $\sum b_n x^n$ with the same radii $R_a = R_b = R$ does not necessarily have the same radius of convergence $R_c = R$. e.g. $\sum a_n x^n = \sum(1/n! - 1/n)x^n$, $\sum b_n x^n = \sum(1/n! + 1/n)x^n$, $\sum c_n x^n = \sum(1/n!)x^n$ so $R_a = R_b = 1$ but $R_c = \infty$. You can however use the convergence of $\sum a_n x^n$, $\sum b_n x^n$ for $|x| < R$ to prove that $\sum c_n x^n$ converges for $|x| < R$. Additionally, if $\sum c_n x^n$ diverges for one of the endpoints $x = R$ or $x = -R$, then this would prove that $R_c = R$.