Math 1B Section 101
07 Oct 2009 Quiz #6 (15 min)

Name: ____________________________________ Score: /10

Determine whether the series converges or diverges. Explain.

1. (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\sin n}}{n^2}$

Ans:

Since $\sin n \leq 1$ for all n, we have $\dfrac{e^{\sin n}}{n^2} \leq \dfrac{e^1}{n^2}$, so $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\sin n}}{n^2} \leq \sum_{n=1}^{\infty} \frac{e}{n^2}$.

Now, $\displaystyle\sum_{n=1}^{\infty} \frac{e}{n^2} = e \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges according to the p-test because $p = 2 > 1$.

Thus, by the Standard Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\sin n}}{n^2}$ converges.

2. (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$

Ans 1:

We do a Limit Comparison Test with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{1/\sqrt{n^2+4}}{1/n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+4/n^2}} = 1 > 0$$

Thus, because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the original series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$ diverges.

Ans 2 (with thanks to Yu-Jen):

We do a Standard Comparison Test with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+2}$.

$$\frac{1}{n+2} = \frac{1}{\sqrt{n^2+4n+4}} \leq \frac{1}{\sqrt{n^2+4}}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+2} = \sum_{n=3}^{\infty} \frac{1}{n}$ diverges because it is a harmonic series.

Thus, the original series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$ diverges.

[Please turn over.]

3. (2 points) $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$ [Hint: Telescoping]

Ans 1:
Since $\ln(1+\frac{1}{n}) = \ln\frac{n+1}{n} = \ln(n+1) - \ln n$, the N-th partial sum is

$$\begin{aligned} s_N &= \sum_{n=1}^{N} \ln(1+\frac{1}{n}) \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(N+1) - \ln N) \\ &= \ln(N+1) - \ln 1 = \ln(N+1) \end{aligned}$$

Thus, $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n}) = \lim_{n\to\infty} s_N = \infty$, so the series diverges.

Ans 2 (with thanks to Fionna, Nancy, Yu-Jen):
We do a Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$. By L'Hospital's Rule,

$$\lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = \lim_{n\to\infty} \frac{\big(1/(1+1/n)\big)(-1/n^2)}{(-1/n^2)} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series and diverges, the original series $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$ diverges.

4. (bonus, 0 points) $\sum_{n=1}^{\infty} (e^{1/n} - 1)$

Method 1:
We use the inequality $1 + x < e^x$. This gives $\frac{1}{n} < e^{\frac{1}{n}} - 1$.
Using the Standard Comparison Test, this shows that the series diverges.

Method 2:
We use the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n} = \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1 \quad \text{[by L'Hospital's rule]}$$

Hence, the series diverges.

[End of Quiz]

Quiz Statistics

Score	Count
0	1
1	0
2	2
3	1
4	2
5	6
6	5
7	5
8	4
9	2
10	2

Ave = 6.00

Key to Comments

B Brilliant with Big picture
You can see what the solution is straight away, and you know which theorems to apply.

C Clear Conceptual understanding of class material
You understand the tools taught in class, how they should be used, where they should be used, and you remember the exact hypotheses to all the theorems and formulas.

R Rigorous mathematically
You understand the logic behind your arguments. No loopholes can escape you. You demand full understanding and proofs of all the theorems and formulas taught in class.

A Arithmetically detailed and careful
You will never make a sign mistake or lose a scaling constant or forget a substitution. Every step is carefully written to avoid careless mistakes.

E Explanations are clear and complete
You explain all the steps in your solution, stating which theorems and formulas are used, leaving no room for doubt. Your solutions are neat and well-presented.

Grading Scheme

Q1)
(1 pt) idea of comparing with $1/n^2$.
(1 pt) correct inequality with e/n^2.
(1 pt) stating Standard Comparison Test or writing $\sum e^{\sin n}/n^2 \leq \sum e/n^2$ (note $\sum$ sign).
(1 pt) stating $\sum 1/n^2$ by p-series test.

Q2)
(1 pt) idea of comparing with $1/n$.
(1 pt) correct evaluation of $\lim n/\sqrt{n^2+4}$
(1 pt) stating Limit Comparison Test.
(1 pt) stating $\sum 1/n$ diverges by p-series test or harmonic series.

Q3)
(1 pt) idea of writing $\ln(1+1/n)$ as $\ln(n+1) - \ln n$.
(1 pt) correcting summing telescoping series and showing limit diverges.

Common Mistakes

Q1 Tried using Integral Test but failed to notice $e^{\sin x}/x^2$ is not decreasing.
Q1 Tried using Limit Convergence Test but $\lim e^{\sin n}$ DNE.
Q1 Wrong inequality $e^{\sin n}/n^2 < 1/n^2$.
Q1 Wrong conclusion with Standard Comparison Test: e.g. less than a diverging series.
Q2 Wrong inequality $1/n \leq 1/\sqrt{n^2+4}$.
Q2 Not knowing how to prove $\lim n/\sqrt{n^2+4} = 1$.
Q1,2 Not stating the test used.
Q1,2 Not stating why $\sum 1/n^2$, $\sum 1/n$ converges, diverges.
Q1,2 Saying a_n diverges rather than $\sum a_n$ diverges.
Q3 "$\ln(1/n) \leq \ln(1+1/n)$ and $\sum \ln(1/n)$ diverges. By the SCT, $\sum \ln(1+1/n)$ diverges."
The SCT cannot be used here because the terms of $\sum \ln(1/n)$ are negative.
Q3 Writing the correct telescoping series but summing the terms wrongly.