Math 1B Section 101
23 Sep 2009 Quiz #4 (15 min)

Name: ______________________________ Score: /10

1. (5 points) Find the area of the surface obtained by rotating about the x-axis the curve $x = 1 + y^2, 0 \le y \le 1$.

   Ans:

$$\begin{aligned} \text{Surface Area} &= \int_0^1 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy \\ &= \int_0^1 2\pi y \sqrt{1 + 4y^2}\ dy \end{aligned}$$

   We make a substitution $u = 1 + 4y^2, du = 8y\ dy$.

$$\text{Surface Area} = \frac{2\pi}{8} \int_1^5 \sqrt{u}\ du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_1^5 = \frac{\pi}{6}(5\sqrt{5} - 1)$$

2. Consider the region bounded by the curves $y = 1 - x^2$ and $y = 0$.
   (a) (2 points) Find the area of the region.
   (b) (3 points) Find the coordinates of the centroid of the region.

   Ans:
   (a)

$$A = \int_{-1}^1 1 - x^2\ dx = 2\int_0^1 1 - x^2\ dx = 2\left[x - \frac{1}{3}x^3\right]_0^1 = \frac{4}{3}.$$

   (b) Because the region is symmetric about the y-axis, we have $\bar{x} = 0$. Meanwhile,

$$\begin{aligned} \bar{y} &= \frac{1}{A}\int_{-1}^1 \frac{1}{2}(1 - x^2)^2\ dx \\ &= \frac{1}{A} \cdot 2\int_0^1 \frac{1}{2}(1 - 2x^2 + x^4)\ dx \\ &= \frac{1}{A}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 \\ &= \frac{3}{4} \cdot \frac{8}{15} = \frac{2}{5} \end{aligned}$$

   In this question, noticing the symmetry about the y-axis helped simplify the computations. We replace integrals in the interval $[-1, 1]$ with twice the integrals in $[0, 1]$.

[Please turn over.]

3. (bonus, 0 points) Consider a curve $y = f(x)$, $a \le x \le b$. Let $s(x)$ be the arc length function. We thicken the curve by a small amount δ perpendicular to the curve. Breaking this region into small pieces of length ds and width δ, we see that its centroid is

$$\bar{x} = \frac{\int x\delta\ ds}{\int \delta\ ds} = \frac{\int x\ ds}{\int ds}, \quad \bar{y} = \frac{\int y\delta\ ds}{\int \delta\ ds} = \frac{\int y\ ds}{\int ds}$$

This formula is only an approximation, but as δ goes to 0, it becomes exact.

Show that the area of the surface obtained by rotating the curve about the x-axis is the product of the arc length of the curve with the distance traveled by its centroid. Use this to find the surface area of a donut, formed by rotating a circle of radius r centered at $(x, y) = (0, R)$ about the x-axis, where $R > r$. What about rotating a square of width $2r$ centered at $(0, R)$ with sides parallel to the x, y-axes?

Ans: The area of the surface of revolution is $\int 2\pi y\ ds$, while the product of the arc length with the distance travelled by the centroid is

$$\left(\int ds\right) \cdot \left(2\pi\bar{y}\right) = \left(\int ds\right) \cdot \left(2\pi \frac{\int y\ ds}{\int ds}\right) = \int 2\pi y\ ds$$

as required. The centroid of a circle is its center, so the distance travelled by the center of the above circle is $2\pi R$. The arc length of the circle is its circumference $2\pi r$. Hence, the surface area of the donut is $4\pi^2 Rr$.

The surface area of the rotated square by the centroid formula is $(2\pi R)(8r) = 16\pi Rr$. One can check that this agrees with the answer derived by summing up the areas of the surfaces obtained by rotating each side of the square.

$$\begin{aligned} \text{top} &= 2\pi(R + r)(2r) \\ \text{bottom} &= 2\pi(R - r)(2r) \\ \text{left} = \text{right} &= \pi((R + r)^2 - (R - r)^2) = 4\pi Rr \end{aligned}$$

[End of Quiz]

Quiz Statistics

Score	Count
0	2
1	0
2	0
3	1
4	2
5	2
6	3
7	2
8	5
9	4
10	9

Ave = 7.30

Common Mistakes

Q1 Using $\int 2\pi x \, ds$ rather than $\int 2\pi y \, ds$
Q2 Simplification errors: the computation is correct but the final answer wrong.
Q3 Wrong region (only half the correct one).
Q3 Wrong formula for centroid.
Q3 Switching the formulas for $\bar{x}$ and $\bar{y}$.
Q3 Not noticing that the region is symmetric about the y-axis, so $\bar{x} = 0$.