Math 1B Section 101
16 Sep 2009 Quiz #3 (15 min)

Name: ______________________________ Score: /10

1. (2 points) Using Simpson's Rule with $n = 4$, approximate the integral $\int_0^1 24x^2 \sin x\pi \, dx$. [Assume $\sin \frac{\pi}{4} \approx 0.71$.]

Ans: Let $f(x) = 24x^2 \sin x\pi$. Using Simpson's Rule,

$$\begin{aligned}
\int_0^1 24x^2 \sin x\pi \, dx &\approx \frac{1/4}{3}\Big[f(0) + 4f(\frac{1}{4}) + 2f(\frac{1}{2}) + 4f(\frac{3}{4}) + f(1)\Big] \\
&= \frac{24}{12}\Big[0 + 4(\frac{1}{4})^2 \sin\frac{\pi}{4} + 2(\frac{1}{2})^2 + 4(\frac{3}{4})^2 \sin\frac{\pi}{4} + 0\Big] \\
&= 5\sin\frac{\pi}{4} + 1 \\
&\approx 4.55
\end{aligned}$$

[Afternote: There was an error in the quiz, where we had $24x^2 \sin x$ instead of $24x^2 \sin x\pi$. To make up for loss of time, full credit is given for writing down the formula for Simpson's rule correctly.]

2. (3 points) How large should n be to guarantee that the Trapezium Rule approximation of $\int_1^2 e^{1/x} \, dx$ is accurate to within 0.0001? [Use $|E_T| \le \frac{K(b-a)^3}{12n^2}$ and $\sqrt{e} \approx 1.65$.]

Ans: Let $f(x) = e^{1/x}$. First, we need to find $K = \max_{1\le x\le 2} |f''(x)|$.

$$\begin{aligned}
f'(x) &= -\frac{1}{x^2}e^{1/x} \\
f''(x) &= \frac{2}{x^3}e^{1/x} + \frac{1}{x^4}e^{1/x}
\end{aligned}$$

Since $\frac{2}{x^3}$, $\frac{1}{x^4}$ and $e^{1/x}$ are decreasing functions, $f''(x)$ is maximal at $x = 1$, so

$$|f''(x)| < 3e = K.$$

We want to find n satisfying

$$\begin{aligned}
0.0001 &\ge \frac{3e(2-1)^3}{12n^2} \\
n &\ge \sqrt{\frac{e}{4(0.0001)}} = \frac{100\sqrt{e}}{2} \approx 84.5
\end{aligned}$$

Therefore, n should be at least 85.

[Please turn over.]

3. (3 points) Is the integral $\int_1^\infty \frac{1}{x^2+2x-8}\,dx$ convergent or divergent? Explain.

Heuristics: We first try substituting $x = 1$ and $x = \infty$ into the integrand, and find that the first is negative and the second is positive. Something funny must be happening in the middle. We try factorizing the denominator and indeed, it is $(x-2)(x+4)$. This suggests partial fractions. Also, mentally, we know we need to check the integral near $x = 2^-$, $x = 2^+$ and $x = \infty$. The third one is convergent, because the integrand grows like $1/x^2$. The first and second are divergent, because the integrand grows like $1/(x-2)$ (since $1/(x+4)$ is bounded) and so the integral behaves like $\ln|x-2|$.

Ans: By partial fractions,

$$\begin{aligned} \frac{1}{x^2+2x-8} &= \frac{1}{(x-2)(x+4)} = \frac{1}{6}(\frac{1}{x-2} - \frac{1}{x+4}) \\ \int \frac{1}{x^2+2x-8}\,dx &= \frac{1}{6}\Big(\ln|x-2| - \ln|x+4|\Big) + C \end{aligned}$$

Because of the discontinuity at $x = 2$, our improper integral equals

$$\int_1^\infty \frac{1}{x^2+2x-8}\,dx = \int_1^2 + \int_2^3 + \int_3^\infty \frac{1}{x^2+2x-8}\,dx$$

The first integral in the above sum is

$$\int_1^2 \frac{1}{x^2+2x-8}\,dx \quad = \quad \frac{1}{6}\lim_{t\to 2^-}\Big[\ln|x-2| - \ln|x+4|\Big]_1^t = -\infty$$

Therefore, our original integral is divergent.

4. (2 points) Is the integral $\int_2^\infty \frac{\sqrt{x}\sin^4 x}{1+x^9}\,dx$ convergent or divergent? Explain.

Heuristics: We first note that the term $\sin^4 x \le 1$ is bounded so it won't give us any problems. The problem is reduced to studying $\sqrt{x}/(1+x^9)$. We quickly check that this function does not have any discontinuities in $[2,\infty)$. Thus, we only need to check what happens near $x = \infty$. Indeed, there, the function grows like $1/x^{8.5}$ which is convergent.

Ans: Since $\sin^4 x \le 1$, we have the bound

$$\frac{\sqrt{x}\sin^4 x}{1+x^9} < \frac{\sqrt{x}}{1+x^9} < \frac{\sqrt{x}}{x^9} = x^{-8.5}$$

Hence, by the Comparison Theorem,

$$\int_2^\infty \frac{\sqrt{x}\sin^4 x}{1+x^9}\,dx < \int_2^\infty x^{-8.5}\,dx = \lim_{t\to\infty}\Big[-\frac{1}{7.5}x^{-7.5}\Big]_2^t = \frac{1}{7.5}2^{-7.5}$$

Therefore, our original integral is convergent.

Quiz Statistics

Score	Count
0	2
1	0
2	0
3	0
4	2
5	5
6	9
7	5
8	6
9	1
10	0

Ave = 5.97

Common Mistakes

Q2. Differentiating $1/x$ gives $-1/x^2$, not $\ln x$.

Q2. If we want to ensure that $|E_T| < 0.0001$, we should have

$$|E_T| = \frac{K(b-a)^3}{12n^2} < 0.0001 \quad \text{and not} \quad 0.0001 < \frac{K(b-a)^3}{12n^2}.$$

Q3. Factorizing $x^2 + 2x - 8$ gives $(x+4)(x-2)$, not $(x-4)(x+2)$.

Q3. Some people did a comparison test using

$$\frac{1}{x^2+2x-8} < \frac{1}{x^2}.$$

This inequality is not correct for small x (try $x = 3$).
Some did a comparison test using

$$\frac{1}{x^2+2x-8} > \frac{1}{x^2+2x}.$$

This inequality IS correct. And $\int_1^\infty \frac{1}{x^2+2x}\, dx$ is convergent. But this does not show that $\int_1^\infty \frac{1}{x^2+2x-8}\, dx$ is convergent because the inequality is on the wrong side.

Q4. A lot of people ended up with

$$\frac{\sqrt{x}\sin^4 x}{1+x^9} \le \frac{\sqrt{x}}{1+x^9},$$

a good first step. Sadly, the latter fraction is hard to integrate (use $u = \sqrt{x}$?).
If we take it one step further,

$$\frac{\sqrt{x}}{1+x^9} < \frac{\sqrt{x}}{x^9} = x^{-8.5},$$

that would have done the trick :)

[End of Quiz]