Math 1B Section 101
02 Dec 2009 Quiz #14 (15 min)

Name: ______________________________ Score: /10

Find the general solution of the differential equation $y'' - 2y' + 2y = 2e^x \cos x$ using:

1. (5 points) the method of undetermined coefficients.

complimentary equation $y'' - 2y' + 2y = 0$

aux equation $r^2 - 2r + 2 = 0$

$r = 1 \pm i$

complimentary solutions $y = C_1 e^x \cos x + C_2 e^x \sin x$

trial solution $y = Axe^x \cos x + Bxe^x \sin x$

Here, we multiplied the trial solution by x because $e^x \cos x$ and $e^x \sin x$ are complimentary solutions

$$y' = A\begin{pmatrix} e^x\cos x \\ +xe^x\cos x \\ -xe^x \sin x\end{pmatrix} + B\begin{pmatrix} e^x\sin x \\ +xe^x\sin x \\ +xe^x\cos x\end{pmatrix}$$

$$= (Ax + Bx + A)e^x \cos x + (Bx - Ax + B)e^x \sin x$$

$$y'' = (A+B)e^x\cos x + (Ax + Bx + A)(e^x\cos x - e^x \sin x)$$
$$+ (B-A)e^x \sin x + (Bx - Ax + B)(e^x \sin x + e^x \cos x)$$
$$= 2(Bx + A + B)e^x \cos x + 2(-Ax - A + B)e^x \sin x$$

$\therefore$ $2e^x\cos x = y'' - 2y' + 2y = 2Be^x \cos x - 2Ae^x \sin x$

Hence, $B = 1$, $A = 0$.

The general solution is

$$y(x) = xe^x \sin x + C_1 e^x \cos x + C_2 e^x \sin x.$$

[Please turn over.]

2. (5 points) the method of variation of parameters.

complimentary solution $y = c_1 e^x \cos x + c_2 e^x \sin x$

particular solution $y = u_1 e^x \cos x + u_2 e^x \sin x$

subject to conditions

$$u_1' e^x \cos x + u_2' e^x \sin x = 0 . \quad (1)$$

$$u_1' \begin{pmatrix} e^x \cos x \\ -e^x \sin x \end{pmatrix} + u_2' \begin{pmatrix} e^x \sin x \\ + e^x \cos x \end{pmatrix} = 2e^x \cos x \quad (2)$$

Since e^x is positive, we divide (1), (2) by e^x:

$$u_1' \cos x + u_2' \sin x = 0 \quad (3)$$

$$u_1' (\cos x - \sin x) + u_2' (\sin x + \cos x) = 2\cos x \quad (4)$$

Subtracting (4) − (3):

$$-u_1' \sin x + u_2' \cos x = 2\cos x \quad (5)$$

Now, (3) $\sin x$ + (5) $\cos x$ gives:

$$u_2' (\sin^2 x + \cos^2 x) = 2\cos^2 x$$

$$\Rightarrow u_2' = 2\cos^2 x = \cos 2x + 1 \quad (6)$$

$$\Rightarrow u_2 = \int \cos 2x + 1 \, dx = \frac{1}{2} \sin 2x + x + C_2$$

From (3) and (6),

$$u_1' = -\frac{u_2' \sin x}{\cos x} = -2\cos x \sin x = -\sin 2x$$

$$\Rightarrow u_1 = \int -\sin 2x \, dx = \frac{1}{2} \cos 2x + C_1$$

Thus, the general solution is

$$y(x) = \left(\frac{1}{2} \cos 2x + C_1\right) e^x \cos x + \left(\frac{1}{2} \sin 2x + x + C_2\right) e^x \sin x$$

Note that $\left(\frac{1}{2} \cos 2x\right) e^x \cos x + \left(\frac{1}{2} \sin 2x\right) e^x \sin x$

$$= \frac{1}{2} e^x (\cos 2x \cos x + \sin 2x \sin x) = \frac{1}{2} e^x \cos(2x - x)$$

$$= \frac{1}{2} e^x \cos x$$

so we may absorb this term into $C_1 e^x \cos x$, giving

$$y(x) = C_1 e^x \cos x + C_2 e^x \sin x + x e^x \sin x .$$

Quiz Statistics

Scores	0	1	2	3	4	5	6	7	8	9	10
	5	0	0	2	3	4	9	3	4	0	0

Average 4.83

Grading Scheme

Q1)
(1 pt) Correct auxiliary equation and roots.
(1 pt) Correct complimentary solution y_c.
(1 pt) Correct form of trial solution y_p.
(1 pt) Correct solving of coefficients in trial solution.
(1 pt) Writing the general solution as $y = y_c + y_p$.

Q2)
(1 pt) Correct form of particular solution.
(1 pt) Correct conditions that u_1, u_2 must satisfy.
(1 pt) Solving and integrating for u_1.
(1 pt) Solving and integrating for u_2.
(1 pt) Writing the general solution with constants of integration.

Observations

Q1. Forgetting to multiply the trial solution by x.
Q1. Arithmetic mistakes in differentiating and summing y_p, y_p', y_p''.
Q2. In $e^x(c_1 \cos x + c_2 \sin x)$, saying $y_1 = \cos x, y_2 = \sin x$
instead of $y_1 = e^x \cos x, y_2 = e^x \sin x$.
Q2. The student painstakingly derives the conditions
$u_1' y_1 + u_2' y_2 = 0, u_1' y_1' + u_2' y_2' = G(x)/a$
rather than just recalling and applying it.