Math 1B Section 101
18 Nov 2009 Quiz #12 (15 min)

Name: ______________________________ Score: /10

1. (6 points) Solve the initial value problem.

$$(\cos x)\, y' - (\sin x)\, y = -\sin x, \quad y(0) = 1, \quad 0 < x < \frac{\pi}{2}.$$

[Hint: What is $(y\cos x)'$?]

[Remark: This equation can also be written as the separable equation $\dfrac{y'}{y-1} = \tan x$.]

Note that $(y\cos x)' = (\cos x)\, y' - (\sin x)\, y$

Thus, the equation becomes

$$(y\cos x)' = -\sin x$$

$$y\cos x = \cos x + C \quad \text{where } C \text{ is any real number.}$$

$$y = 1 + C/\cos x$$

When $x=0$, we have $y=1$. Thus

$$1 = 1 + C/\cos 0 = 1 + C$$

Therefore, $C=0$ and we have the constant solution

$$y = 1.$$

Alternatively, the equation $y' = (y-1)\tan x$ tells us that the equilibrium solution is $y=1$. Since $y(0)=1$, we must have $y(x)=1$ for all $0 < x < \pi/2$.

[Please turn over.]

2. In the homework, we showed that if populations of rabbits and wolves satisfy

$$\frac{dR}{dt} = 0.08R - 0.001RW, \quad \frac{dW}{dt} = -0.02W + 0.00002RW,$$

then the phase trajectories satisfy

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW} = \frac{-rW + bRW}{kR - aRW}$$

which has solutions of the following form. Here, C is a constant.

$$\frac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$$

(a) (2 points) Find the equilibrium solutions (there are two of them).

The equilibrium solutions satisfy

$$\frac{dR}{dt} = R(0.08 - 0.001W) = 0, \quad \frac{dW}{dt} = W(-0.02 + 0.00002R) = 0$$

If $R = 0$, then $\frac{dR}{dt} = 0$, and $\frac{dW}{dt} = -0.02W = 0$ implies $W = 0$.

If $R \neq 0$, then $\frac{dR}{dt} = 0$ implies $W = \frac{0.08}{0.001} = 80$ and $\frac{dW}{dt} = 80(-0.02 + 0.00002R) = 0$ implies $R = \frac{0.02}{0.00002} = 1000$. Thus, $(R, W) = (0, 0)$ or $(1000, 80)$.

(b) (2 points) What is the value of C at these solutions?

At $(R, W) = 0$, $C = \frac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = 0$

At $(R, W) = (1000, 80)$, $C = \frac{1000^{0.02}\, 80^{0.08}}{e^{0.00002(1000)}\, e^{0.001(80)}} = \left(\frac{1000}{e}\right)^{0.02}\left(\frac{80}{e}\right)^{0.08}$

(c) (bonus, 0 points) Show that for all phase trajectories, C lies between these values.

Note that $C = \left(\frac{R^r}{e^{aR}}\right)\left(\frac{W^k}{e^{bW}}\right)$. The function $f(R) = R^r/e^{aR}$

is maximized at $f'(R) = \frac{(rR^{r-1})(e^{aR}) - (ae^{aR})(R^r)}{(e^{aR})^2} = 0$

$\Rightarrow R^{r-1}e^{aR}(r - aR) = 0 \Rightarrow R = r/a$. Similarly, $\frac{W^k}{e^{bW}}$ is maximized at $W = k/b$. Hence, C is maximum at the equilibrium $(R, W) = (r/a, \frac{k}{b})$.

Finally, $R, W \geq 0 \Rightarrow C \geq 0$.

Quiz Statistics

Scores	0	1	2	3	4	5	6	7	8	9	10
	2	0	1	2	1	4	3	7	6	2	2

Average 6.17

Grading Scheme

Q1)
either:
(2 pts) Finding integrating factor.
(2 pts) Correct integration after multiplying by factor.
(2 pts) Solving for C.
or:
(4 pts) Solving equation using $(y\cos x)'$.
(2 pts) Solving for C.
or:
(4 pts) Solving separable equation $y'/(y+1) = \tan x$.
(2 pts) Solving for C.

Q2a)
(1 pt) Solving $dW/dt = dR/dt = 0$.
(1 pt) Correctly getting both solutions $(R, W) = (0, 0)$ and $(R, W) = (1000, 80)$.

Q2b)
(2 pt) Correctly substituting both solutions above to get $C = 0$ and $C = (1000/e)^{0.02}(80/e)^{0.08}$.

Observations

Q1. Forgetting $+C$, or forgetting to solve for C using initial conditions.
Q1. Not knowing the integral of $\tan x$ for the integrating factor.
Q1. Forgetting the negative sign in the integrating factor $e^{\int -\tan x dx}$.
Q1. Multiplying the integrating factor $\cos x$ to the original equation $(\cos x)y' - (\sin x)y = -\sin x$ rather than to $y' - (\tan x)y = -\tan x$.
Q2. Solving $dW/dR = 0$ rather than $dW/dt = 0, dR/dt = 0$ for the equilibrium solutions.
Q2. Forgetting the equilibrium solution $(R, W) = (0, 0)$.
Q2. The student solves $dW/dt = 0, dR/dt = 0$ and gets "$R = 0$ or 1000, $W = 0$ or 80" but fails to pair up the values correctly to get $(R, W) = (0, 0)$ and $(R, W) = (1000, 80)$. Indeed, for instance, one can check that $(R, W) = (0, 80)$ is not an equilibrium solution because $dW/dt \neq 0$.