Math 1B Section 101
18 Nov 2009 Quiz #11 (15 min)

Name: ________________________ Score: /10

Billy has a serious roach problem. He estimates there are about 20 roaches in his room. Diabolically, he figures that if he catches 15 every month, his problem will be solved. His Math 1B GSI suggests that he test his plan using a logistic population model. Billy studies his roaches and comes up with the following equation:

$$\frac{dP}{dt} = 2P(1 - \frac{P}{40}) - 15$$

where P is the number of roaches and t is time measured in months.

1. (2 points) What are the equilibrium solutions to the above differential equation?

At the equilibrium solutions, $\frac{dP}{dt} = 2P(1-P/40) - 15 = 0$.

$$-\frac{2}{40}(P^2 - 40P + \frac{15(40)}{2}) = 0$$

$$-\frac{1}{20}(P - 40P + 300) = 0$$

$$-\frac{1}{20}(P-30)(P-10) = 0$$

Thus, the equilibrium solutions are $P = 10$ and $P = 30$.

2. (5 points) By making a substitution $y = P - 10$, show that

$$\frac{dy}{dt} = y(1 - \frac{y}{20}).$$

Use this to solve the original equation above with the initial condition $P(0) = 20$.

$$\frac{dP}{dt} = 2P(1 - \frac{P}{40}) - 15 = -\frac{1}{20}(P-30)(P-10).$$

Substituting $P = y + 10$ and $dP/dt = dy/dt$, we get

$$\frac{dy}{dt} = -\frac{1}{20}(y-20)(y) = y(1 - \frac{y}{20}).$$

This is the logistic model with $k = 1$, $K = 20$, and initial condition $y(0) = P(0) - 10 = 10$.

[Please turn over.]

Thus, the solution is

$$y = \frac{K}{1+Ae^{-kt}} = \frac{20}{1+Ae^{-t}}, \qquad A = \frac{K-y_0}{y_0} = \frac{20-10}{10} = 1$$

In terms of P, we get

$$P(t) - 10 = \frac{20}{1+\ e^{-t}} \Rightarrow P(t) = \frac{20}{1+\ e^{-t}} + 10$$

3. (1 points) Can Billy eradicate the revolting roaches? Explain.

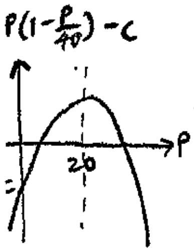
As $t \to \infty$, $\quad P(t) \to \frac{20}{1+0} + 10 = 30.$

Thus, Billy will not be able to eradicate the roaches.

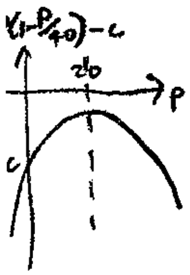
4. (2 points) To solve his problem, at least how many roaches must he catch monthly? [Hint: At time $t = 0$, we should have $\frac{dP}{dt} < 0$.]

Let c be the number of roaches he must catch.
The equation becomes

$$\frac{dP}{dt} = 2P\left(1-\frac{P}{40}\right) - c$$



If $\frac{dP}{dt} > 0$ at $t=0$, then P increases but cannot increase indefinitely because $2P(1-P/40)-c$ is negative for large P. Thus, P will taper off at an equilibrium solution greater than $P_0 = 20$.



If $\frac{dP}{dt} < 0$ at $t=0$, then P decreases and $\frac{dP}{dt}$ decreases farther. Thus, eventually $P = 0$. Solving $\left.\frac{dP}{dt}\right|_{t=0} = 2P_0\left(1-\frac{P_0}{40}\right) - c < 0$ gives $c > 2P_0\left(1-\frac{P_0}{40}\right) = 2(20)\left(1-\frac{20}{40}\right) = 20$. Hence, $c \geqslant 21$.

[End of Quiz]

Quiz Statistics

Scores											
	0	1	2	3	4	5	6	7	8	9	10
	2	0	1	1	4	7	5	3	3	2	2

Average 5.63

Grading Scheme

Q1)
(1 pt) stating equilibrium is $dP/dt = 0$.
(1 pt) solving quadratic equation to get $P = 10, 30$.

Q2)
(2 pt) correctly showing that $dy/dt = y(1 - y/20)$.
(2 pt) solving the differential equation in y.
(1 pt) using $P = y + 10$ to get the solution in P.

Q3)
(1 pt) saying as $t \to \infty$, $P \to 30$, or that $dP/dt > 0$ at $t = 0$.

Q4)
(1 pt) writing $dP/dt = 2P(1 - P/40) - c$ where c is number of roaches to catch.
(1 pt) solving $2P_0(1 - P_0/40) - c < 0$, giving $c > 20$.

Observations

1. Many students are not aware that for differential equations, equilibrium solutions are the constant solutions. To find the equilibrium solutions, we set $dP/dt = 0$ and solve for P as a constant.

2. Some students failed to recognize the logistic equation $dy/dt = y(1 - y/20)$ and went on to solve this separable equation using partial fractions which is a lot of extra work.

3. Some students thought that the equation $dP/dt = 2P(1 - P/20) - 15$ is a logistic equation, and incorrectly applies the formula to solve it. Rather, this is a logistic model *with harvesting*.

4. The aim of this quiz is to show that the logistic model *with harvesting* can be solved using the usual formula for the logistic equation using a change of variable. The change of variable is $y = P - E_0$ where E_0 is the lower equilibrium solution.'