Math 1B Section 101
BIG IDEAS
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This set of notes is not a complete summary of the course material. They are just some comments coming out of my looking at the homework problems and quizzes.

7 Broad Overview

7.1 Integration by parts

7.2 Trigonometric Integrals

a. $\displaystyle\int \sin^m x \cos^n x \, dx$

b. $\displaystyle\int \tan^m x \sec^n x \, dx$

c. $\displaystyle\int \sin mx \cos nx \, dx$

7.3 Trigonometric Substitution

a. Basic forms $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$ and $\sqrt{x^2 - a^2}$
Basic substitutions $x = \sin\theta$, $x = \tan\theta$ and $x = \sec\theta$

b. Converting trig functions of θ into functions of x using right-angled triangles

7.4 Integration of Rational Functions by Partial Fractions

a. Knowing how to do partial fraction decomposition

b. Knowing how to integrate:

i. $\displaystyle\int \frac{1}{(ax+b)^k} \, dx, \; k \geq 1$

ii. $\displaystyle\int \frac{1}{x^2 + a^2} \, dx$

iii. $\displaystyle\int \frac{x}{x^2 + a^2} \, dx$

7.5 Strategy for Integration

a. Simplify the integrand if possible

b. Look for an obvious substitution

c. Classify the integrand according to its form

7.1. Integration by parts
7.2. Trigonometric functions
7.3. Radicals
7.4. Rational functions

d. Try again.

7.7 Approximate Integration

a. Midpoint Rule
b. Trapezium Rule
c. Simpson's Rule
d. Overestimate or underestimate? Larger or smaller?
e. Error bounds for each rule

7.8 Improper Integrals

a. Type I: involves ∞ or $-\infty$
b. Type 2: involves discontinuity at a point c
c. Evaluating improper integrals using limits
d. Evaluate $\int_{-\infty}^{\infty}$ by $\int_{-\infty}^{0} + \int_{0}^{\infty}$ or any other midpoint
e. Evaluate $\int_{a}^{b}$ by $\int_{a}^{c} + \int_{c}^{b}$ if there is a discontinuity at c
f. Convergence, divergence of improper integrals
g. Comparison theorem

8 Broad Overview

8.1 Arc Length

a. Formula for arc length
b. Arc length function $s(x)$ for a given starting point

8.2 Area of a Surface of Revolution

a. Formula for area of rotation about x-axis
b. Formula for area of rotation about y-axis

8.3 Applications to Physics and Engineering

a. Hydrostatic force
b. Formula for moments around x-axis, y-axis
c. Formula for centroid of a region
d. Theorem of Pappus for volume of revolution

11 Broad Overview

11.1 Sequences

a. Definitions
   i. Limit of sequence
   ii. Convergent, divergent sequences
   iii. $\lim a_n = \infty$
   iv. Increasing, decreasing, monotonic sequences
   v. Bounded above, bounded below, bounded
   vi. $a_n \to L$ as $n \to \infty$

b. Evaluation of limits of sequences
   i. Sum, difference, product, quotient, scaling, taking powers
   ii. If $\lim f(x) = L$ and $f(n) = a_n$, then $\lim a_n = L$.
   iii. If $\lim a_n = L$ and f is continuous at L, then $\lim f(a_n) = f(L)$.
   iv. If $\lim |a_n| = 0$, then $\lim a_n = 0$.
   v. Squeeze theorem
   vi. Monotone convergent theorem

c. Geometric sequences
   i. $\{r^n\}$ is convergent for $-1 < r \leq 1$, divergent otherwise.
   ii. $\lim r^n = 0$ for $|r| < 1$ and $\lim r^n = 1$ for $r = 1$.

11.2 Series

a. Definitions
   i. Series, partial sum s_n, sum s of a series
   ii. Convergent, divergent series

b. Evaluation of series
   i. If $\sum a_n$ is convergent, then $\lim a_n = 0$.
      If $\lim a_n$ does not exist, or $\lim a_n \neq 0$, then $\sum a_n$ is divergent.
   ii. If $\sum a_n$, $\sum b_n$ are convergent, so are $\sum ca_n$, $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$.

c. Geometric series
   i. $\sum ar^{n-1}$ convergent if $|r| < 1$, divergent otherwise.
   ii. $\sum_{n=1}^{\infty} ar^{n-1} = a/(1-r)$ for $|r| < 1$.

11.3 The Integral Test and Estimates of Sums

$a_n = f(n)$, f continuous, positive, decreasing on $[1, \infty)$.

a. Integral test: $\sum a_n$ convergent $\Leftrightarrow \int_1^{\infty} f(x)\,dx$ convergent
b. Remainder estimate: $\int_{n+1}^{\infty} f(x)\,dx \leq s - s_n \leq \int_n^{\infty} f(x)\,dx$
c. Bounds on sum: $s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_n^{\infty} f(x)\,dx$
d. p-Test: $\sum 1/n^p$ convergent for $p > 1$, divergent otherwise.

Student Questions

Q: Are we required to know things like the reduction formula for sine on a test?
A: No, but you should know how to derive it.

Q: Q37 on p482 requires you to integrate $1/(x^2 + 4x + 6)^2$. Must we know this?
A: No, it is not part of the course. So please disregard that problem for the homework.

Q: Are we required to know error bound formula for approximate integration?
A: No, you won't have to memorize those formulas (but may be asked to use them, in which case they would be provided to them on the exam). You should know the definitions for K in the formula, e.g. $K \geq |f''(x)|$ for the Midpoint and Trapezium rules.

Midterm Questions

Q: What sections are we tested for Midterm 1?
A: Sections 7.1 to 11.3.

Q: What formulas will be provided for the midterm?
A: Error bounds for approximate integration, trig formulas in the red box on page 465, specific reduction formulas (Section 7.1) (you may be given such a formula, though, and be asked to apply it or to derive it), and the values of density ρ of water and the gravitational constant g. You will be required to know the integrals on pg 484, except for 15, 16 (hyperbolic trigonometric functions).

Q: Can we bring our own scrap paper for the midterm?
A: No. Paper will be provided upon request.

7.1 Integration by parts

Important derivatives (memorize them):
$\frac{d}{dx}x^n = nx^{n-1}$
$\frac{d}{dx}\ln x = \frac{1}{x}$, but be aware of the absolute value sign in $\int \frac{1}{x}\, dx = \ln|x| + C$.
$\frac{d}{dx}e^x = e^x$
$\frac{d}{dx}\sin x = \cos x$
$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

Some guidelines for picking the parts:

- If $\ln x$ appears in the integral, differentiate that part. (Q1)
- If e^x appears in the integral, integrate that part.
- If it is a trig function, either differentiation or integration is fine.
- If it is the inverse of a trig function, differentiate it.

Useful techniques:

- Make a substitution before integrating by parts. (Q29, Q33)
- Make a substitution after integrating by parts. (Q10).
- Repeated integration by parts. (Q43)
- If you get the original integral after integrating by parts a few times, rearrange the resulting equation. (Q17, Q47).
- Often more than one way to break apart an integral.
  (Q15: $u = \ln x, dv = \ln x\ dx$ or $u = (\ln x)^2, dv = dx$.)

Common mistakes:

- forgetting the factor $\frac{1}{5}$ in $dv = \cos 5x\ dx, v = \frac{1}{5}\sin 5x$. (Q3)
- writing $uv + \int v\ du$ instead of $uv - \int v\ du$.
- forgetting $+C$ in indefinite integrals.
- writing $+C$ for definite integrals!

7.2 Trigonometric Integration

Important formula (memorize them):

Derivatives and integrals:
$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\sec x = \sec x \tan x$
$\int \tan x\ dx = \ln|\sec x| + C$ remember the absolute value signs!
$\int \sec x\ dx = \ln|\sec x + \tan x| + C$ remember the absolute value signs!

Pythagorean identities:
$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$

Half-angle Identities:
$\cos 2x = 2\cos^2 x - 1$
$\cos 2x = 1 - 2\sin^2 x$
$\sin 2x = 2\sin x \cos x$

How to evaluate $\int \sin^m x \cos^n x\ dx$:

1.

Extract a factor of Write everything else in terms of using the identity Then, make a substitution	$\sin x$ $\cos x$ $\sin^2 x = 1 - \cos^2 x$ $u = \cos x$	$\cos x$ $\sin x$ $\cos^2 x = 1 - \sin^2 x$ $u = \sin x$
This works for	$m = 2k+1, n \in \mathbb{R}$	$m \in \mathbb{R}, n = 2k+1$

2. If Step 1 is unsuccessful (i.e. $\int \sin^{2k} x \cos^{2l} x\ dx$):
   Use the half-angle identities. Expand. Return to Step 1.

How to evaluate $\int \tan^m x \sec^n x\ dx$:

1.

Extract a factor of Write everything else in terms of using the identity Then, make a substitution	$\sec^2 x$ $\tan x$ $\sec^2 x = 1 + \tan^2 x$ $u = \tan x$	$\sec x \tan x$ $\sec x$ $\tan^2 x = \sec^2 x - 1$ $u = \sec x$
This works for	$m \in \mathbb{R}, n = 2k \geq 2$	$m = 2k+1, n \in \mathbb{R}$

2. If Step 1 is unsuccessful:
   (a) For $\int \tan^{2k} x \sec^n x\ dx$, ($k \neq 0$, $n = 0$ or odd)
   Write $\tan^{2k} x = (\sec^2 x - 1)^k$. Expand. Return to Step 1.
   (b) For $\int \sec^{2l+1} x\ dx$,
   Write $u = \sec^{2l-1} x, dv = \sec^2 x$. Integrate by parts. (See below.)

Mantra:
If power of $\sec x$ is positive even, pull out a $\sec^2 x$.
If power of $\tan x$ is positive odd, pull out a $\sec x \tan x$.
Otherwise, convert all $\tan\theta$ to $\sec\theta$, and go to the start!
Only the odd pure $\sec\theta$ is stubborn. Kill him by parts!

An important trigonometric example $\int \sec^{2n+1} x \; dx$:

$$\begin{array}{ll} u = \sec^{2n-1} x & dv = \sec^2 x \; dx \\ du = (2n-1)(\sec^{2n-2} x)(\sec x \tan x) \; dx & v = \tan x \end{array}$$

$$\begin{aligned} \int \sec^{2n+1} x \; dx &= \sec^{2n-1} x \tan x - (2n-1) \int \sec^{2n-1} x \tan^2 x \; dx \\ &= \sec^{2n-1} x \tan x - (2n-1) \int \sec^{2n-1} x (\sec^2 x - 1) \; dx \\ \int \sec^{2n+1} x \; dx &= \frac{1}{2n} \sec^{2n-1} x \tan x + \frac{2n-1}{2n} \int \sec^{2n-1} x \; dx \end{aligned}$$

How to evaluate $\int \sin mx \sin nx \; dx$, $\int \sin mx \cos nx \; dx$, $\int \cos mx \cos nx \; dx$:
Use the identities (given in exam)

$$\begin{aligned} \sin A \cos B &= \frac{1}{2}[\sin(A-B) + \sin(A+B)] \\ \sin A \sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)] \\ \cos A \cos B &= \frac{1}{2}[\cos(A-B) + \cos(A+B)] \end{aligned}$$

7.3 Trigonometric substitution

When integrating functions containing $\sqrt{f(x)}$ where $f(x)$ is quadratic:

1. Check if the substitution $u = f(x)$ works.
2. If not, complete the square for $f(x)$.

Expression	$\sqrt{a^2 - x^2}$	$\sqrt{x^2 + a^2}$	$\sqrt{x^2 - a^2}$
Pythagorean identity	$1 - \sin^2 x = \cos^2 x$	$\tan^2 x + 1 = \sec^2 x$	$\sec^2 x - 1 = \tan^2 x$
Substitution	$x = \sin\theta$	$x = \tan\theta$	$x = \sec\theta$

3. Find a trig substitution based on the Pythagorean identities.
4. After the substitution, we usually get a trigonometric integral like those in Section 7.2.
5. Remember to convert the θ's back to x's at the end, using a right-angle triangle.

Useful Techniques and Reminders:

- Remember to substitute $dx = f'(\theta)d\theta$ when you substitute $x = f(\theta)$.
- Complete the square if necessary. (Q27)
- Know how to draw a right-angle triangle to compute, say, $\sin\theta$ from $\sec\theta$.
- Remember SOH CAH TOA.
- Recognize when quadratic substitution is better than trigonometric substitution! (Q25)
- Make a substitution to convert polynomial under the square root to a quadratic. (Q29)
- Remember the range (e.g. $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$) where the substitution is valid!
  In general, for any substitution, one should check that
  it is one-to-one in the interval we are integrating over.

7.4 Integration by Partial Fractions

The main steps are:

1. Write as partial fractions.
   (a) Do long division so that the numerator has a smaller degree than the denominator.
   (b) Factorize the denominator into linear and quadratic terms.
   (c) Determine what terms appear in the partial fraction decomposition. (see below).
   (d) Solve for the unknowns in the terms.
2. Integrate each term separately.
   (a) Complete the square for quadratic denominators.
   (b) Apply substitutions where necessary.

Terms appearing in the partial fraction decomposition:
Factorize the denominator into linear and quadratic terms. A simple theorem in algebra says that this is always possible (although sometimes the coefficients may look very ugly). Suppose one of the linear factors in the denominator is $(ax+b)^k$. Then, the partial fractions for this linear factor is

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

where $A_1, A_2, \ldots, A_k$ are some real numbers. If the quadratic factor $(ax^2+bx+c)^k$ appears in the denominator, then the partial fractions for this factor looks like

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

where $A_1, B_1, A_2, B_2, \ldots, A_k, B_k$ are real numbers.

(P.S. That simple theorem in algebra says that the roots of a real polynomial are either real or occur in complex conjugate pairs.)

Some important identities (easily derived):

- $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$
- $\displaystyle\int \frac{1}{(ax+b)^k}\,dx$, substitute $u = ax+b$.
- $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
- $\displaystyle\int \frac{1}{(x^2+a^2)^k}\,dx$ (not in syllabus), substitute $x = \tan\theta$.
- $\displaystyle\int \frac{x}{(x^2+a^2)^k}\,dx$, substitute $u = x^2+a^2$.

7.5 Strategy for Integration

Sometimes, we have to keep playing with the integral before we get the answer. The slick solutions we show for homework and quizzes often hide all the guesswork that goes on behind the scenes. Use lots of scrap paper, lots of trial and error. Don't feel that you need to write out the answer on your first try.

Don't be discouraged if you failed on your first attempt. Study what you did, and try to understand what went wrong. It will usually give you a hint of other approaches to try.

BONUS: An interesting substitution from the homework is Section 7.4, Q57-61. Its called the Weierstrauss substitution $t = \tan(x/2)$ and it is for rational functions of $\sin x$ and $\cos x$. Under this substitution,

$$\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2}{1+t^2}dt$$

If you can't find your integral and are really desperate, you can always fall back on this.

7.7 Approximate Integration

Formulas for Midpoint, Trapezoidal and Simpson's Rules (memorize them):

- $\displaystyle\int_a^b f(x)\,dx \approx M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$
- $\displaystyle\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$
- $\displaystyle\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)]$

Errors for these approximations (will be given in exams,
except for definition of K in terms of f'' and $f^{(4)}$):

- $\displaystyle |E_M| \le \frac{K(b-a)^3}{24n^2}, \quad |f''(x)| \le K$ for $a \le x \le b$
- $\displaystyle |E_T| \le \frac{K(b-a)^3}{12n^2}, \quad |f''(x)| \le K$ for $a \le x \le b$
- $\displaystyle |E_S| \le \frac{K(b-a)^5}{180n^4}, \quad |f^{(4)}(x)| \le K$ for $a \le x \le b$

Comments:

- Know how to check visually if the approximation is an overestimate or underestimate.
- Know how to check visually how to rank the different estimates according to size. (Q1d)
- When using the approximations, we need to find the maximum of $|g(x)|$ where $g(x) = f''(x)$ or $f^{(4)}(x)$. This can be done using methods from Math 1A, checking if the function $g(x)$ is increasing or decreasing, or by solving $g'(x) = 0$ to find the maximum and minimum of the function. (Q22)

7.8 Improper Integrals

Summary:

- Improper integral of type 1: involves ∞ or $-\infty$.
- Improper integral of type 2: involves discontinuity at $x = c$.
- Convergence, divergence of improper integrals
- Definition of $\int_{-\infty}^{\infty}$ and $\int_a^b$ (with discontinuity at c where $a < c < b$).
- Comparison theorem (for checking convergence, divergence).
- $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ but diverges for $p \le 1$.

Comments:

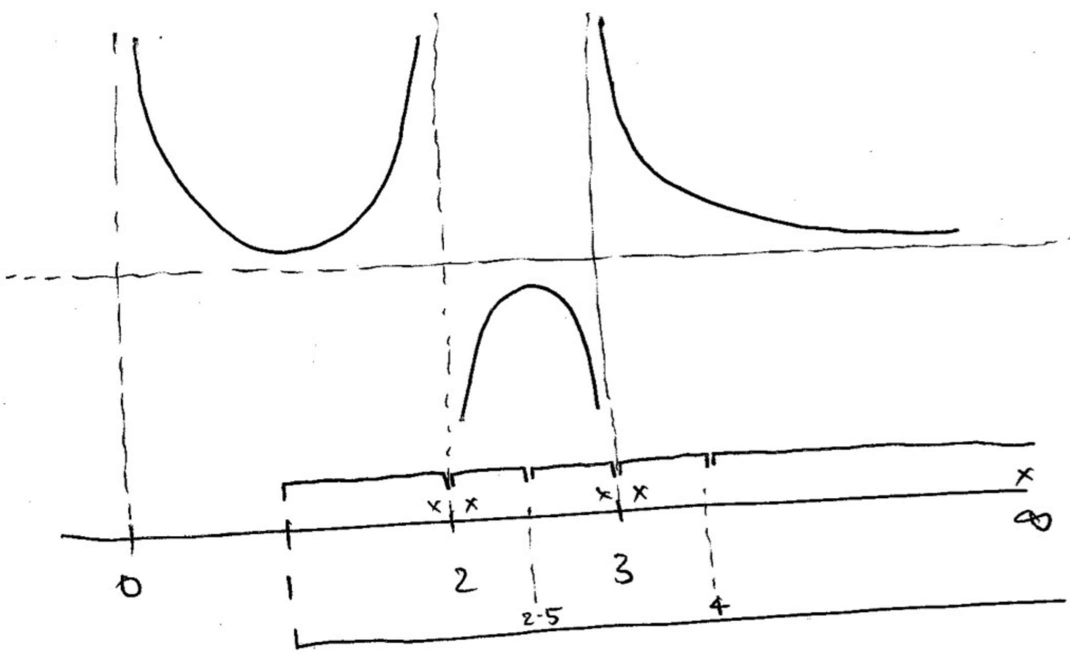
- Check for ALL discontinuities in the range of integration. (Q7)
- Suppose $\int f(x)\,dx = F(x) + C$. To compute $\int_{-\infty}^{\infty} f(x)\,dx$, do not evaluate $[F(x)]_{-\infty}^{\infty}$! Remember to pick a midpoint, say $x = 0$, so that
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to-\infty}[F(x)]_t^0 + \lim_{t\to\infty}[F(x)]_0^t.$$
- For $p > 1$, $\int_1^\infty \frac{1}{x^p}\,dx$ converges but $\int_0^1 \frac{1}{x^p}\,dx$ diverges! (Note the range of integration.)
- If the graph is negative, then the area under the graph is also negative. (Q13) We apply the Comparison Theorem to the ABSOLUTE VALUE of the integrand.
- Q49 is a very typical analysis of the convergence of an improper integral using the Comparison theorem. In
$$\int_0^\infty \frac{x}{x^3+1}\,dx,$$
the integrand and integral is well behaved near for small x. We only wonder what will happen for x large. Namely, we wonder if
$$\int_{1000000}^\infty \frac{x}{x^3+1}\,dx$$
converges (replace 1000000 with your favorite large number). The integrand grows roughly like $x/x^3 = 1/x^2$ whose integral converges. We must mimic this in our use of the comparison theorem. So we use
$$\frac{x}{x^3+1} \le \frac{x}{x^3} = \frac{1}{x^2}.$$
The rest of the problem is easy. Note that if we did not restrict ourselves to large x, say $x > 1000000$, then $\int \frac{1}{x^2}\,dx$ is NOT convergent at $x = 0$ and we could not use the comparison test! (see also Q53)
- Sometimes, you are asked to integrate something that has p as a parameter (see Q59). Be careful not to over-generalize and do something like
$$\int x^p\,dx = \frac{1}{p+1}x^{p+1} + C$$
and ignore the different cases, say $p = -1$ and $p < -1$ and $-1 < p$. The above formula is not valid for $p = -1$!

Evaluating Improper Integrals

1. Find the bad points.
2. Break up the integral.
3. Express each part as a limit.

Example: $\displaystyle\int_1^\infty f(x)\ dx, \quad f(x) = \frac{e^{-x^2}}{x(x-2)(x-3)}$

The bad points are at $x = 0, 2, 3, \infty$. We break up the interval $[1, \infty)$ so that each smaller interval has only one bad point as an endpoint.



$$\begin{aligned}\int_1^\infty f(x)\ dx &= \int_1^2 + \int_2^{2.5} + \int_{2.5}^3 + \int_3^4 + \int_4^\infty f(x)\ dx \\ &= \lim_{t\to 2^-}\int_1^t f(x)\ dx + \lim_{t\to 2^+}\int_t^{2.5} f(x)\ dx + \cdots + \lim_{t\to\infty}\int_4^t f(x)\ dx\end{aligned}$$

We could have picked any other middle points besides $x = 2.5$ and $x = 4$.

Convergence and Divergence

1. Study the integral NEAR each bad point.
2. If the integral diverges near one bad point, the whole integral diverges.

The main idea here is that we study a SMALL NEIGHBORHOOD of the bad point.

Example 1. Consider $\int_0^\infty \frac{x}{x^3+1}\,dx$. We could use the inequality

$$\frac{x}{x^3+1} \le \frac{x}{x^3} = \frac{1}{x^2}.$$

Unfortunately,

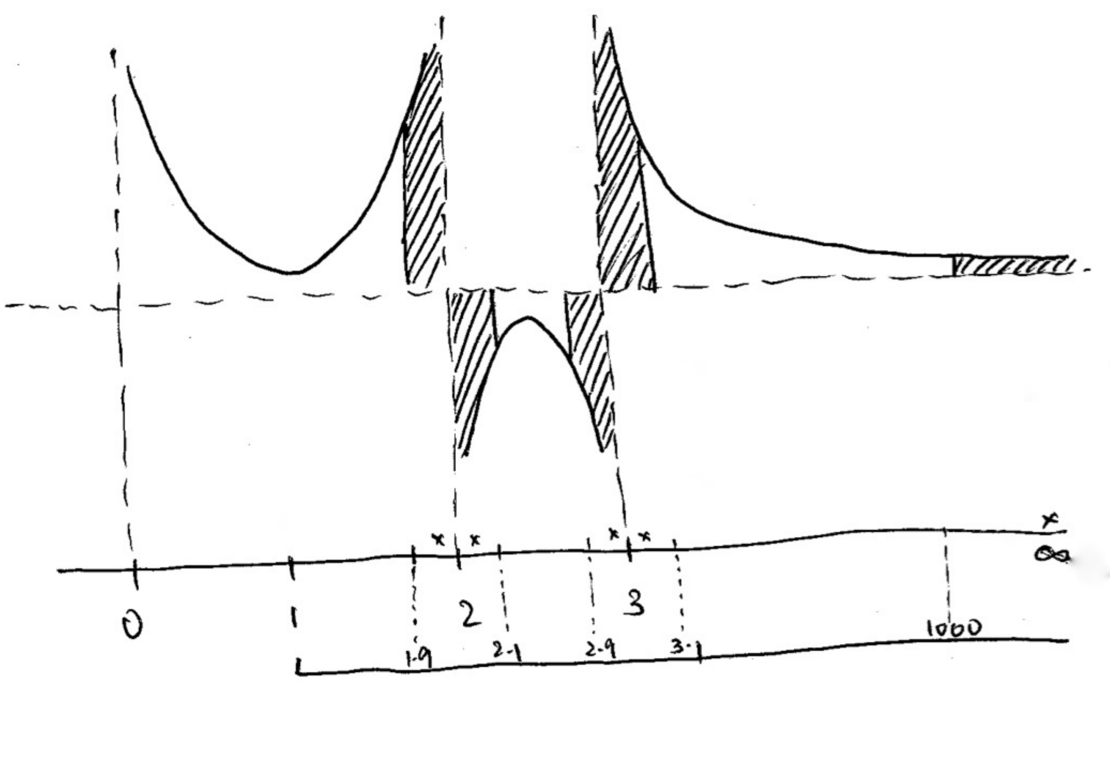
$$\int_0^\infty \frac{1}{x^2}\,dx = \int_0^1 \frac{1}{x^2}\,dx + \int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to 0}\Big[-\frac{1}{x}\Big]_t^1 + \int_1^\infty \frac{1}{x^2}\,dx$$

and the first limit goes to ∞, so the integral diverges. This does not tell us anything about the convergence or divergence of $\int_0^\infty \frac{x}{x^3+1}\,dx$ because the inequality is on the wrong side. However, if we write

$$\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^{1000} \frac{x}{x^3+1}\,dx + \int_{1000}^\infty \frac{x}{x^3+1}\,dx,$$

the first integral is proper so it converges. The second integral can be compared with $\int_{1000}^\infty \frac{1}{x^2}\,dx$ which converges. Hence, the original integral converges. This is because we studied a "SMALL NEIGHBORHOOD" of the bad point $x = \infty$.

Example 2. Consider again $\displaystyle\int_1^\infty f(x)\,dx, \quad f(x) = \frac{e^{-x^2}}{x(x-2)(x-3)}.$



As seen before, the bad points are at $x = 2, 3, \infty$. This time we break up the integral into

$$\int_1^\infty f(x)\ dx = \int_1^{1.9} + \int_{1.9}^{2} + \int_2^{2.1} + \int_{2.1}^{2.9} + \int_{2.9}^{3} + \int_3^{3.1} + \int_{3.1}^{1000} + \int_{1000}^{\infty} f(x)\ dx$$

In the intervals $[1, 1.9]$, $[2.1, 2.9]$ and $[3.1, 1000]$, the integral is proper and thus convergent. We only need to study the other terms which are improper integrals over a small neighborhood of the bad points. Take for example

$$\int_3^{3.1} \frac{e^{-x^2}}{x(x-2)(x-3)}\ dx.$$

<u>MENTALLY</u>, we see that for $3 \leq x \leq 3.1$,

$$e^{-x^2} \approx e^{-3^2}, \quad \frac{1}{x} \approx \frac{1}{3}, \quad \frac{1}{x-2} \approx \frac{1}{3-2} = 1$$

so we approximate

$$\int_3^{3.1} \frac{e^{-x^2}}{x(x-2)(x-3)}\ dx \approx \frac{e^{-3^2}}{(3)(1)} \int_3^{3.1} \frac{1}{x-3}\ dx = \frac{e^{-3^2}}{(3)(1)} \lim_{t \to 3} \Big[\ln|x-3| \Big]_t^{3.1}$$

and the last limit diverges. This hints to us that our original integral diverges too. Let's write out the solution formally, using the Comparison Theorem.

<u>FORMALLY</u>, for $3 \leq x \leq 3.1$,

$$e^{-3.1^2} \leq e^{-x^2} \leq e^{-3^2}, \quad \frac{1}{3.1} \leq \frac{1}{x} \leq \frac{1}{3}, \quad \frac{1}{1.1} \leq \frac{1}{x-2} \leq \frac{1}{1}.$$

Therefore,

$$\frac{e^{-3.1^2}}{(3.1)(1.1)} \frac{1}{x-3} \leq \frac{e^{-x^2}}{x(x-2)(x-3)}$$

However,

$$\int_3^{3.1} \frac{1}{x-3}\ dx = \lim_{t \to 3} \Big[\ln|x-3| \Big]_t^{3.1} = \infty$$

Thus, by the Comparison Theorem, $\int_3^{3.1} \frac{e^{-x^2}}{x(x-2)(x-3)}\ dx$ diverges and hence $\int_1^\infty \frac{e^{-x^2}}{x(x-2)(x-3)}\ dx$ also diverges.

Heuristically, when studying the integral near a bad point, we isolate the part of the function that behaves badly, and bound all the other parts. Depending on whether the integral of the bad part of the function is divergent or convergent, the original integral will also behave likewise.

Limit Comparison Test for Improper Integrals

A very useful tool for checking if an improper integral converges or diverges is the Limit Comparison Test. It is not in the textbook or syllabus, so please check with the professor or GSI before using it in an exam.

Theorem (Limit Comparison Test)

Suppose

1. $f(x) > 0, g(x) > 0$ for all $x \in [a, \infty)$.
2. f, g are continuous on $[a, \infty)$.
3. $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = L$

Then,

- If $0 < L < \infty$, then $\int_a^\infty f(x)\ dx$ converges $\Leftrightarrow \int_a^\infty g(x)\ dx$ converges.
- If $L = 0$, then $\int_a^\infty f(x)\ dx$ converges $\Leftarrow \int_a^\infty g(x)\ dx$ converges.
- If $L = \infty$, then $\int_a^\infty f(x)\ dx$ converges $\Rightarrow \int_a^\infty g(x)\ dx$ converges.

The theorem is also true when we replace
"$[a, \infty)$, $\lim\limits_{x \to \infty}$, $\displaystyle\int_a^\infty$" with "$[a, b)$, $\lim\limits_{x \to b^-}$, $\displaystyle\int_a^b$" or "$(a, b]$, $\lim\limits_{x \to a^+}$, $\displaystyle\int_a^b$".

8.1 Arc Length, 8.2 Area of a Surface of Revolution

- Know the arc length formula. Sometimes, dy/dx can be computed implicitly.
- Know what is an arc length function with a given point as the starting point.
- Given $\sqrt{f(x)}$, try to write $f(x) = g(x)^2$ before integrating it by brute force. (Q11)
- Know the two formula (rotating around x-axis or y-axis) for area of surface of revolution.

One easy way to remember these formulas is

$$(ds)^2 = (dx)^2 + (dy)^2$$

where ds may be thought of as the length of a small arc along the curve.
Arc length becomes

$$\int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{1 + (\frac{dy}{dx})^2}\ dx = \int \sqrt{1 + (\frac{dx}{dy})^2}\ dy$$

Suppose we have a curve given by $y = f(x)$ or $x = g(y)$.
Area of surface of revolution around the $x-axis$ (so y is the radius of revolution) becomes

$$\begin{aligned} \int 2\pi y\ ds &= \int 2\pi y \sqrt{1 + (\frac{dy}{dx})^2}\ dx = \int 2\pi f(x) \sqrt{1 + f'(x)^2}\ dx \\ &= \int 2\pi y \sqrt{1 + (\frac{dx}{dy})^2}\ dy = \int 2\pi y \sqrt{1 + g'(y)^2}\ dy \end{aligned}$$

Area of surface of revolution around the $y-axis$ (so x is the radius of revolution) becomes

$$\begin{aligned} \int 2\pi x\ ds &= \int 2\pi x \sqrt{1 + (\frac{dy}{dx})^2}\ dx = \int 2\pi x \sqrt{1 + f'(x)^2}\ dx \\ &= \int 2\pi x \sqrt{1 + (\frac{dx}{dy})^2}\ dy = \int 2\pi g(y) \sqrt{1 + g'(y)^2}\ dy \end{aligned}$$

I listed many formula above because I wanted to show that the important ones to remember are really just $\int 2\pi y\ ds$ and $\int 2\pi x\ ds$.

Common mistakes:
(a) Check if range of integration $\int_a^b$ is given in terms of the x domain or the y domain. Make sure that this matches with the dx or dy at the end of the integrand.
(b) Check if the integrand has the term $2\pi x$ or $2\pi y$ in it. This depends only on which axis we are rotating around, and not on the range of integration or the dx, dy at the end of the integrand!

8.3 Applications to Physics and Engineering

- Know how to compute the hydrostatic force on a surface from first principles.
- Know how to compute the moment around the x-axis or y-axis of a region
- Know how to compute the centroid of a region
- Know the Theorem of Pappus for computing the volume of revolution.

Side-note: There is a similar Pappus Theorem for the area of a surface of revolution.
It says that the area is the length of the curve multiplied with the distance travelled by the centroid *of that curve* under the revolution. (see S8.2Q33)

Cool tricks:

1. Moments about an axis can be added or subtracted like areas or volumes! (Q40-41)
2. The centroid of a triangle is the average of the coordinates of the vertices (not necessarily true of other polygons).

Common mistakes:

- Getting the units wrong by a power of 10.
- Note that the formula

$$M_y = \rho \int_a^b x f(x)\ dx, \quad M_x = \rho \int_a^b \frac{1}{2} f(x)^2\ dx$$

  are for a region bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$.

  Use Cool Trick #1 to find moments for other regions, such as that between two curves $y = f(x)$ and $y = g(x)$ where $f(x) > g(x) > 0$. (Q31)

  In this case, the moment about the y-axis is

$$\rho \int_a^b x f(x)\ dx - \rho \int_a^b x g(x)\ dx$$

  and the moment around the x-axis is

$$\rho \int_a^b \frac{1}{2} f(x)^2\ dx - \rho \int_a^b \frac{1}{2} g(x)^2\ dx.$$

11.1 Sequences

- Formal definition of the limit of a sequence
- Convergent, divergent sequences
- Definition of $\lim_{n\to\infty} a_n = \infty$
- Definition of increasing, decreasing, monotonic
- Definition of bounded above, bounded below, bounded sequence.

(1) If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n\to\infty} a_n = L$.
(2) If $\{a_n\}$ is convergent, we can scale, take powers or apply a continuous function to it.
If $\{b_n\}$ is also convergent, we can add, substract, multiply, divide (if $\lim b_n \neq 0$) them.
(3) Squeeze theorem: $a_n \leq b_n \leq c_n$, $\lim a_n = \lim c_n = L$ implies $\lim b_n = L$.
(4) Every bounded, monotonic sequence is convergent.

- If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.
- $\{r^n\}$ is convergent for $-1 < r \leq 1$ and divergent otherwise.
$\lim r^n = 0$ for $|r| < 1$ and $\lim r^n = 1$ for $r = 1$.

Notes:
About (2): it is possible that a sequence $\{a_n\}$ has a limit even though $f(x)$ does not.
e.g. $f(x) = \sin(x\pi)$, $a_n = f(n)$, $\lim_{n\to\infty} a_n = 0$.
So "the limit of $f(x)$ does not exist" doesn't prove that $\{a_n\}$ is divergent. (Q27)

Q69 from the textbook:
Given $a_1 = 1, a_{n+1} = 3 - 1/a_n$, show that it is bounded above, and monotone increasing.

Bounded above:
Claim: $a_n \leq 3$ for all n.
Proof by induction:
$a_1 = 1 \leq 3 \quad \surd$
Now, assume that $a_n \leq 3$ for some n.
We want to show that $a_{n+1} \leq 3$, i.e. $3 - 1/a_n \leq 3$.
Indeed, $a_n \leq 3 \Rightarrow 1/a_n \geq 1/3 \Rightarrow -1/a_n \leq -1/3 \Rightarrow 3 - 1/a_n \leq 3 - 1/3 < 3 \quad \surd$

Monotone increasing:
Claim: $a_n \leq a_{n+1}$ for all n.
Proof by induction:
$a_1 = 1 < 2 = a_2 \quad \surd$
Now, assume that $a_n \leq a_{n+1}$ for some n.
We want to show that $a_{n+1} \leq a_{n+2}$, i.e. $3 - 1/a_n \leq 3 - 1/a_{n+1}$.
Indeed, $a_n \leq a_{n+1} \Rightarrow 1/a_n \geq 1/a_{n+1} \Rightarrow -1/a_n \leq -1/a_{n+1} \Rightarrow 3 - 1/a_n \leq 3 - 1/a_{n+1} \quad \surd$

11.2 Series

- Series, partial sum, sum of a series. Convergent, diverg ent series.
- Geometric series $\sum ar^{n-1}$ convergent if $|r| < 1$, divergent otherwise. Sum $a/(1-r)$.
- If $\sum a_n$ is convergent, then $\lim a_n = 0$.
  If $\lim a_n$ does not exist, or $\lim a_n \neq 0$, then $\sum a_n$ is divergent.
- If $\sum a_n$, $\sum b_n$ are convergent, then so are $\sum ca_n$, $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$.

Useful: The sum of a convergent series and a divergent series is divergent. (Q69)

11.3 The Integral Test and Estimates of Sums

- $\sum 1/n^p$ is convergent if $p > 1$, divergent otherwise.

Let $f(x)$ be continuous, positive, decreasing on $[1, \infty)$, $a_n = f(n)$.

- **Integral Test**: $\sum a_n$ is convergent iff $\int_1^\infty f(x)\,dx$ is convergent.
- **Remainder Estimate**: If $\sum a_n$ convergent, $R_n = s - s_n$, then

$$\int_{n+1}^\infty f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx$$

- **Bounds**:

$$s_n + \int_{n+1}^\infty f(x)\,dx \le s \le s_n + \int_n^\infty f(x)\,dx$$

Q33 from the textbook:

(a) The sum of the first 10 terms is $s_{10} = 1.54976773$. To see how good this estimate is, we consider how far it is from the actual sum s. According to the formula (2) on pg 701, the remainder $R_{10} = s - s_{10}$ satisfies

$$\int_{11}^\infty \frac{1}{x^2}\,dx \le R_{10} \le \int_{10}^\infty \frac{1}{x^2}\,dx = \Big[-\frac{1}{x}\Big]_{10}^\infty = \frac{1}{10}$$

Thus, our estimate $s_{10} = 1.54976773$ is at most 0.1 away from the actual sum s.

(b) Using the formula (3) on pg 702, we have

$$\begin{aligned} s_{10} + \int_{11}^\infty \frac{1}{x^2}\,dx \quad &\le s \le \quad s_{10} + \int_{10}^\infty \frac{1}{x^2}\,dx = \Big[-\frac{1}{x}\Big]_{10}^\infty = \frac{1}{10} \\ s_{10} + \frac{1}{11} \quad &\le s \le \quad s_{10} + \frac{1}{10} \\ 1.64067682 \quad &\le s \le \quad 1.64976773 \end{aligned}$$

Taking the midpoint of the two bounds, we get the estimate $s \approx 1.64522228$ with an error of at most $\frac{1}{2}(\frac{1}{10} - \frac{1}{11}) = 0.00454545455$.

Note that in general, we have the estimate

$$s \ \approx \ s_N + \frac{1}{2}\Big(\int_N^\infty f(x)\,dx + \int_{N+1}^\infty f(x)\,dx\Big)$$

with an error of at most

$$\frac{1}{2}\Big(\int_N^{N+1} f(x)\,dx\Big)$$

(c) Solving $R_N \le \displaystyle\int_N^\infty \frac{1}{x^2}\,dx \le 0.001$, we get $\dfrac{1}{N} \le 0.001$ or $N \ge 1000$.

11.4 The Comparison Tests

Standard Comparison Test
Suppose
1. $\sum a_n$, $\sum b_n$ series with positive terms.
2. $a_n \leq b_n$ for all n.
Then,
1. $\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent.
2. $\sum a_n$ divergent $\Rightarrow \sum b_n$ divergent.

Limit Comparison Test
Suppose
1. $\sum a_n$, $\sum b_n$ series with positive terms.
2. $\lim a_n/b_n = L$
Then,
1. If $0 < L < \infty$, then $\sum a_n$ converges $\Leftrightarrow \sum b_n$ converges.
1. If $L = 0$, then $\sum a_n$ converges $\Leftarrow \sum b_n$ converges.
1. If $L = \infty$, then $\sum a_n$ converges $\Rightarrow \sum b_n$ converges.

It is important in this theorem that the terms a_n, b_n are positive. For instance, consider the case where

$$a_n = \frac{1}{n} + (-1)^{n+1}\frac{1}{\sqrt{n}}, \quad b_n = (-1)^{n+1}\frac{1}{\sqrt{n}}.$$

Then, the limit of $a_n/b_n = 1 + (-1)^{n+1}/\sqrt{n}$ as n goes to infinity is 1. $\sum b_n$ is an alternating series that converges, but $\sum a_n = \sum 1/n + \sum(-1)^{n+1}/\sqrt{n}$ is the sum of a divergent series with a convergent series and is thus divergent.

Estimating Sums
Given series $s = \sum a_n$, $t = \sum b_n$ with $a_n < b_n$,
we can get bounds on the remainder

$$s - s_n \ = \ a_n + a_{n+1} + \cdots < b_n + b_{n+1} + \cdots \ = \ t - t_n.$$

11.5 Alternating Series Test

Definition of Alternating Series: $a_n = (-1)^{n-1}b_n$ or $a_n = (-1)^n b_n$ where b_n are positive.

Alternating Series Test Suppose the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies
1. $b_{n+1} \leq b_n \quad$ for all n
2. $\lim_{n\to\infty} b_n = 0$

Then, the series converges.

Alternating Series Estimate Theorem Suppose the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies
1. $b_{n+1} \leq b_n \quad$ for all n
2. $\lim_{n\to\infty} b_n = 0$

Then, $|R_n| = |s - s_n| \leq b_{n+1}$.

11.6 Absolute convergence and the ratio and root tests

Definition of absolute convergence.

Definition of conditional convergence.

Theorem If a series is absolutely convergent, then it is convergent.

Ratio Test
Suppose we have a series $\sum_{n=1}^{\infty} a_n$ satisfying
1. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$
Then
1. If $L < 1$, the series is *absolutely* convergent (therefore convergent).
2. If $L > 1$, the series is divergent.
3. If $L = 1$, the Ratio Test is inconclusive.

Root Test
Suppose we have a series $\sum_{n=1}^{\infty} a_n$ satisfying
1. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$
Then
1. If $L < 1$, the series is *absolutely* convergent (therefore convergent).
2. If $L > 1$, the series is divergent.
3. If $L = 1$, the Root Test is inconclusive.

Rearrangements
1. If $\sum a_n$ is absolutely convergent with sum s,
then any rearrangement of the series has the same sum s.
2. If $\sum a_n$ is conditionally convergent and r is a real number,
then there exists a rearrangement of the series with the sum r.

Useful fact: $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ [See textbook for proof.]

11.7 Strategy for Testing Series

1. $\sum 1/n^p$ [pS]: p-series, convergent if $p > 1$, divergent otherwise.
2. $\sum ar^{n-1}$ [gS]: geometric series, convergent if $|r| < 1$, divergent otherwise.
3. (Standard) Comparison Test [CT]: for series similar to p-series or geometric series.
4. Limit Comparison Test [LCT]: for series similar to p-series or geometric series.
5. Test for Divergence [TFD]: $\lim a_n \neq 0$ implies divergence.
6. Alternating Series Test [AST]: for alternating series.
7. Ratio Test [RaT]: for series involving factorials and products.
8. Root Test [RoT]: for series where a_n is of the form $(b_n)^n$.
9. Integral Test [IT]: for series with decreasing terms.

Test for Divergence If $\lim a_n \neq 0$, then $\sum a_n$ diverges.
Here is a variant, based on the fact that $\lim |a_n| = 0 \iff \lim a_n = 0$.
Test for Divergence If $\lim |a_n| \neq 0$, then $\sum a_n$ diverges.
This variant is good for alternating series.

11.8 Power Series

Definition of *Power Series centered at a*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

The c_n are the *coefficients* of the power series.

Definition of *Radius of Convergence*, R:
The power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for all $|x-a| < R$
and diverges for all $|x-a| > R$.
To find the radius of convergence, we use the Ratio Test (or sometimes the Root Test).

Definition of *Interval of Convergence*:
The set of values of x for which the power series converges. If R is the radius of convergence, then the interval of convergence is one of the following four possibilities:

$$(a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R), \quad [a-R, a+R]$$

To check the end points of the interval, we often use the Alternating Series Test.

Theorem
Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:
1. The series converges only when $x = a$.
2. The series converges for all x.
3. The series has a radius of convergence R for some positive R.

The above theorem is very useful for proving convergence/divergence of power series. For instance, if the series diverges for some $x = b$, then it diverges for all $|x| > |b|$. If the series converges for some $x = b$, then it converges for all $|x| < |b|$. (Q37)

11.9 Representation of Functions as Power Series

Theorem[Term-by-term differentiation and integration]
For a power series centered at a with convergence radius R, the series can be differentiated and integrated term by term in the interval $(a-R, a+R)$, and the resulting series also have convergence radius R.

This theorem is very useful for finding power series expansions of functions that we know are differentials or integrals of easy functions. e.g. $1/(1+x)^2 = d/dx[-1/(1+x)$, $\ln(1-x) = \int -1/(1-x)\ dx$, $\tan^{-1} x = \int 1/(1+x^2)\ dx$

Given $C = \sum c_n x^n = \sum a_n x^n + \sum b_n x^n = A + B$, let the convergence radii be R_a, R_b, R_c.
1. Suppose $R_a = R_b$. Then, $R_a \leq R_c$. But it is possible that $R_a \neq R_c$.
Proof: Pick x such that $|x| < R_a$. Then, A, B converges. So C converges. So the convergence radii of C is at least R_a. But if $A = \sum(1/n! - 1/n)x^n$ and $B = \sum(1/n! + 1/n)x^n$, both $R_a = R_b = 1$ but $C = \sum x^n/n!$ and $R_c = \infty$.
2. Suppose $R_a < R_b$. Then, $R_c = R_a$. The convergence intervals of A, C are the same.
Proof: The above proof also shows $R_a \leq R_c$. Now pick x such that $R_a < |x| < R_b$. Then, A diverges but B converges. So C diverges. Hence $R_c \leq R_a$. Thus $R_a = R_c$. In fact, since B converges for $|x| = R_a$, C converges if and only if A converges, so their interval of convergences are the same.

Be careful with term-by-term differentiation of power series, especially noting what happens to the first term! e.g, differentiating $\sum_{n=0}^{\infty} x^n/n!$ What is wrong with the steps below?

$$\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{nx^{n-1}}{n!} = \sum_{n=0}^{\infty}\frac{x^{n-1}}{(n-1)!}$$

Note that when $n = 0$, the term is $x^{-1}/(-1)!$ which doesn't make sense! We should drop the constant term in the original power series after differentiating.

$$\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} = \frac{d}{dx}\sum_{n=1}^{\infty}\frac{x^n}{n!} = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

11.10 Taylor and Maclaurin Series

Theorem [Taylor Series, Maclaurin Series(a=0)]
If f has a power series expansion at a, i.e. $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, |x-a| < R$, then $c_n = f^{(n)}(a)/n!$.

There are functions not equal to the sum of their Taylor series. e.g. $f(x) = e^{-1/x^2}$ for $x \neq 0$, $f(0) = 0$. These functions do not have power series expansions.

Define $T_n(x) = \sum_{i=0}^{\infty} f^{(i)}(x-a)^i/i!$, $R_n(x) = f(x) - T_n(x)$.
$T_n(x)$ is called the n-th degree Taylor polynomial of f at a.

Theorem
If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series for $|x-a| < R$.

Theorem [Taylor's Inequality]
If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

We use this theorem to prove a function f is equal to its Taylor series, by finding a good bound M_n for $|f^{(n+1)}(x)|$ for each n, and then showing that $M_n/(n+1)! \to 0$ as $n \to \infty$.

Note that the power series of a function around a is a good approximation of the function usually only around a, e.g. the power series $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$ doesn't work well for $x = -1$ even though $1/(1-x) = 1/2$ at $x = -1$. Power series around different points are different.

Multiplication and Division of Power Series.

Using series to approximate limits.

Useful expansions

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \quad \text{for all } -1<x<1 \\
e^x &= \sum_{n=0}^{\infty} x^n/n! \quad \text{for all } x \\
\ln(1-x) &= \sum_{n=1}^{\infty} -\frac{x^n}{n} \quad \text{for all } -1 \le x < 1 \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x \\
\tan^{-1} x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for all } -1 \le x \le 1 \\
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad \text{for all real } k, |x|<1 (\text{endpts??}) \\
\binom{k}{n} &= \frac{k(k-1)\cdots(k-n+1)}{n!}
\end{aligned}
$$

Useful Fact: $\lim_{n\to\infty} x^n/n! = 0$ for all real numbers x.

Useful Fact: $e = 1 + 1/1! + 1/2! + 1/3! + \cdots$.

Applications of Taylor Polynomials

1. Approximating functions by polynomials

Say we want to approximate $f(x_0)$. We want to expand f around some center a. We pick the center a close to x_0 such that $f(a), f'(a), f''(a), \ldots$ are known exactly.

Three ways to approximate the error $|R_n| = |f(x) - T_n(x)|$.
1. Use a graphing calculator.
2. If the series is alternating, use Alternating Series Estimate Theorem.
2. Use Taylor's Inequality

2. Applications to physics

Giving first order, second order, etc. approximations of complicated formulas.