

Unbounded Convex Semialgebraic Sets as Spectrahedral Shadows

Shaowei Lin

9 Dec 2010

Abstract

Recently, Helton and Nie [3] showed that a compact convex semialgebraic set S is a spectrahedral shadow if the boundary of S is nonsingular and has positive curvature. In this paper, we generalize their result to unbounded sets, and also study the effect of the perspective transform on singularities.

1 Introduction

Spectrahedra play a very important role in optimization. Indeed, there are many efficient algorithms for maximizing or minimizing linear cost functions over spectrahedral sets. As a result, a problem of great interest is to determine if a given convex set can be expressed as the projection

$$\{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m, A_0 + \sum_{i=1}^n A_i x_i + \sum_{i=1}^m A_{i+n} u_i \succeq 0\}$$

of a spectrahedra, where the A_i are $d \times d$ matrices for some d . Such projections are known as *spectrahedral shadows*.

Recently, Helton and Nie [3] showed that a compact convex semialgebraic set S is a spectrahedral shadow if the boundary of S is nonsingular and has positive curvature. Because all convex semialgebraic sets have nonnegative curvature at smooth points on their boundaries, the authors went on to conjecture that all compact convex semialgebraic sets are spectrahedral shadows.

There are two main ideas behind their proof. First, they showed that a convex set that is a finite union of spectrahedral shadows is also a spectrahedral shadow. This allowed them reduce the global representation problem to local ones. They then used the moment-type relaxation of Lasserre-Parrilo [2, 4] to construct local representations. It would be interesting to generalize their construction to local neighborhoods of singular points or points with zero curvature on the boundary of a given convex set.

In [5], Nie proposed using perspective transformations as a way of blowing up singularities and removing points with zero curvature on the boundary of the convex set. A perspective transformation $\mathbf{p} : \mathbb{R}_+ \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}_+ \times \mathbb{R}^{n-1}$ is the map $(x_1, x_2, \dots, x_n) \mapsto (1/x_1, x_2/x_1, \dots, x_n/x_1)$. One can show that the perspective transform preserves convexity, is a self-inverse, and maps spectrahedral shadows to spectrahedral shadows. Unfortunately, the transformed set is always unbounded, so the original result of Helton-Nie for compact semialgebraic sets cannot be applied directly. This issue motivated us to generalize their theorem to noncompact sets by studying singularities at infinity.

In this paper, we accomplish this generalization.

Theorem 3.4. *Suppose $S = \cup_{k=1}^m T_k \subset \mathbb{S}^n$ is convex semialgebraic with each*

$$T_k = \{x \in \mathbb{S}^n : g_1^k(x) \geq 0, \dots, g_{m_k}^k(x) \geq 0\}$$

being defined by homogeneous polynomials $g_i^k(x) \geq 0$. If for every $u \in \partial S$, and each g_i^k satisfying $g_i^k(u) = 0$, T_k has interior near u and $g_i^k(x)$ is either sos-concave or strictly quasi-concave at u , then S is a spectrahedral shadow.

The proof of this result follows that of [3, Theorem 3.4] very closely. We also show that the perspective transformation only moves the singularity at the origin to a point at infinity, so the singularity is not simplified. In fact, there are no analytic transformations of \mathbb{R}^n which simplify singularities while preserving the convexity of every convex set. We end this report with a few suggestions on how the Helton-Nie conjecture can be tackled.

2 Linear Transforms of Convex Cones

In this section, we motivate the idea of thinking of convex sets in \mathbb{R}^n as convex cones in \mathbb{R}^{n+1} when studying spectrahedral shadows. In this approach, algebraic inequalities and semidefinite conditions will be homogenized.

A projective transformation is a map $\pi_A : \mathbb{P}^n \rightarrow \mathbb{P}^n$ represented by an invertible $(n+1) \times (n+1)$ matrix A such that

$$\begin{aligned}\pi_A(x_0 : x_1 : \cdots : x_n) &= (y_0 : y_1 : \cdots : y_n), \\ [y_0, y_1, \dots, y_n]^T &= A[x_0, x_1, \dots, x_n]^T.\end{aligned}$$

Such maps take lines to lines. We often view π_A as a transformation of the affine chart $\mathbb{R}^n = \{(1 : x_1 : \cdots : x_n) \in \mathbb{P}^n\}$. The perspective transformation

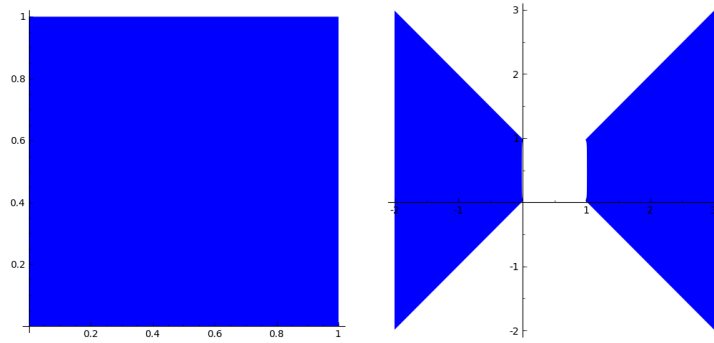
$$\begin{aligned}\mathbf{p} : \mathbb{R}_+ \times \mathbb{R}^{n-1} &\rightarrow \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ (x_1, x_2, \dots, x_n) &\mapsto (1/x_1, x_2/x_1, \dots, x_n/x_1)\end{aligned}$$

is an example of a projective transformation. In projective coordinates, this map takes a point $(x_0 : x_1 : x_2 : \cdots : x_n)$ to the point $(x_1 : x_0 : x_2 : \cdots : x_n)$, so the matrix A representing \mathbf{p} is the identity matrix with the first two rows switched. Note that \mathbf{p} is an involution since $\mathbf{p}^2 = 1$. One can also show that \mathbf{p} takes convex sets to convex sets.

While projective transformations preserve lines, they do not always preserve convexity. For instance, the map π_A represented by

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

takes the square in \mathbb{R}^2 with vertices shown below to an inverted square.



This anomaly occurs because π_A takes the line $x = 1/2$ to the line at infinity. One way to remedy this problem for a given map π_A is to restrict its domain so that convexity is preserved. For example, the domain of the perspective transformation is restricted to the half space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$.

Another solution is to consider the quotient space

$$\mathbb{S}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

where $(x_0, x_1, \dots, x_n) \sim \lambda(x_0, x_1, \dots, x_n)$ for $\lambda > 0$. This space is isomorphic to the n -dimensional sphere, and subsets of \mathbb{S}^n may be thought of as cones in \mathbb{R}^{n+1} . Since there is no ambiguity of confusion with \mathbb{P}^n , we also denote points in \mathbb{S}^n by $(x_0 : x_1 : \dots : x_n)$. Meanwhile, note that the projective space \mathbb{P}^n is given by a similar quotient where in the equivalence relation, λ is allowed to be negative. We now define a *linear transformation* to be a rational map $\pi_A : \mathbb{S}^n \rightarrow \mathbb{S}^n$ represented by a non-zero $(n+1) \times (n+1)$ matrix A such that

$$\begin{aligned} \pi_A(x_0 : x_1 : \dots : x_n) &= (y_0 : y_1 : \dots : y_n), \\ [y_0, y_1, \dots, y_n]^T &= A[x_0, x_1, \dots, x_n]^T. \end{aligned}$$

Here, π_A is rational, because we make no requirement for the matrix A to be invertible so π_A is defined only in the complement of the null space of A . Convex cones in \mathbb{R}^{n+1} are taken to convex cones under this map.

It is tempting to view π_A as a convexity-preserving transformation of

$$\mathbb{R}^n = \{(1 : x_1 : \dots : x_n) \in \mathbb{S}^n\}$$

Unfortunately, this affine chart \mathbb{R}^n is not dense in \mathbb{S}^n because the antipodal points $\{(-1 : x_1 : \dots : x_n) \in \mathbb{S}^n\}$ have been omitted. In fact, this viewpoint is the idea of restricting domains of projective transformations in disguise. It will be helpful not to limit ourselves in this way.

We say that $S \subset \mathbb{S}^n$ is convex if the corresponding cone in \mathbb{R}^{n+1} is convex. A spectrahedron in \mathbb{S}^n is a subset which can be written as

$$\{(x_0 : x_1 : \dots : x_n) \in \mathbb{S}^n \mid A_0 x_0 + A_1 x_1 + \dots + A_n x_n \succeq 0\}$$

while a spectrahedral shadow is a rational projection

$$\{(x_0 : x_1 : \dots : x_n) \in \mathbb{S}^n \mid \exists u \in \mathbb{R}^m, \sum_{i=0}^n A_i x_i + \sum_{i=1}^m A_{i+n} u_i \succeq 0\}.$$

Then, it is clear that spectrahedral shadows are preserved under linear transformations of \mathbb{S}^n , and a subset $S \subset \mathbb{R}^n$ is a spectrahedral shadow in \mathbb{R}^n if and only if the cone

$$\{(1 : x_1 : \dots : x_n) \in \mathbb{S}^n \mid (x_1, \dots, x_n) \in S\}$$

over S is a spectrahedral shadow in \mathbb{S}^n . If S is a semialgebraic set, then the cone over S will be defined by $x_0 \geq 0$ and homogenizations of the algebraic inequalities defining S . Last but not least, the sphere \mathbb{S}^n is compact, so will any semialgebraic subset $S \subset \mathbb{S}^n$. Hence, every open cover of S has a finite subcover. This will be important in proving our main theorem later.

3 Unbounded Spectrahedral Shadows

We begin by reducing the global representation problem to local ones.

Lemma 3.1. *If W_1, \dots, W_m are spectrahedral shadows in \mathbb{S}^n , then the convex hull $\text{conv}(\cup_{k=1}^m W_k)$ is also a spectrahedral shadow in \mathbb{S}^n .*

Proof. Because the W_k are cones in \mathbb{R}^{n+1} , the convex hull of their union is the Minkowski sum $W_1 + \dots + W_m$. The representation of this convex hull as a spectrahedral shadow then follows immediately from the representations of the W_k . See also [3, §2]. Note that [3, Theorem 2.2] was not employed. \square

Given $x, u \in \mathbb{S}^n$, let $d(x, u)$ be the Euclidean distance between x and u as points on the unit sphere in \mathbb{R}^{n+1} . Let $B(u, \delta)$ represent the closed ball $\{x \in \mathbb{S}^n \mid d(x, u) \leq \delta\}$. Following the proof of [3, Proposition 3.2] and using the compactness of \mathbb{S}^n , we have this result.

Proposition 3.2. *Let S be a convex subset of \mathbb{S}^n . Then, S is a spectrahedral shadow if and only if for every $u \in \partial S$, there exists some $\delta > 0$ such that $S \cap \bar{B}(u, \delta)$ is a spectrahedral shadow.*

Now, to solve the local representation problem, we need [2, Theorem 2].

Theorem 3.3. *Suppose $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$ is compact convex and has nonempty interior. If each $g_i(x)$ is either sos-concave or strictly quasi-concave on S , then S is a spectrahedral shadow.*

Next, we clarify the notion of sos-concavity and of quasi-concavity for homogeneous polynomials. Let $u \in \mathbb{R}^{n+1}$ be a point where the k -coordinate u_k is nonzero for some k . We say that a homogeneous polynomial $g(x)$ is sos-concave (or strictly quasi-concave) at u if and only if the dehomogenization

$$g(x_1, \dots, x_{k-1}, u_k, x_{k+1}, \dots, x_n)$$

is sos-concave (or strictly quasi-concave) at

$$(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = (u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n).$$

One can show that this definition is independent of the choice of k . We are now ready for the main theorem.

Theorem 3.4. *Suppose $S = \cup_{k=1}^m T_k \subset \mathbb{S}^n$ is convex semialgebraic with each*

$$T_k = \{x \in \mathbb{S}^n : g_1^k(x) \geq 0, \dots, g_{m_k}^k(x) \geq 0\}$$

being defined by homogeneous polynomials $g_i^k(x) \geq 0$. If for every $u \in \partial S$, and each g_i^k satisfying $g_i^k(u) = 0$, T_k has interior near u and $g_i^k(x)$ is either sos-concave or strictly quasi-concave at u , then S is a spectrahedral shadow.

Proof. Here, the argument is essentially the same as that of [3, Theorem 3.4], except that to represent $T_k \cap \bar{B}(u, \delta)$ as a spectrahedral shadow, we apply a rotation to \mathbb{R}^{n+1} so that u is mapped to $(1, 0, \dots, 0)$ and intersect $T_k \cap \bar{B}(u, \delta)$ with the hyperplane $x_0 = 1$. For sufficiently small δ , the resulting intersection is compact semialgebraic convex in \mathbb{R}^n with nonempty interior and whose defining inequalities are either sos-concave or strictly quasi-concave. Thus, we can apply Theorem 3.3 to get the desired representation. \square

4 Singularities and Perspective Transforms

In singularity theory, the fundamental tool used for simplifying singularities is the blowup map. Let Bl^n be the subset of $\mathbb{R}^n \times \mathbb{P}^{n-1}$ defined by equations $x_i y_j = x_j y_i$ for all i and j , where points in $\mathbb{R}^n \times \mathbb{P}^{n-1}$ are given coordinates $((x_1, \dots, x_n), (y_1 : \dots : y_n))$. Then, the blowup $\rho : \text{Bl}^n \rightarrow \mathbb{R}^n$ is the restriction of the natural projection $\mathbb{R}^n \times \mathbb{P}^{n-1} \rightarrow \mathbb{R}^n$ to Bl^n . Hironaka [1] showed in 1964 that this map is sufficient for resolving singularities completely.

We may cover the manifold Bl^n with n affine charts defined by $y_i = 1$ for $i = 1, \dots, n$. Each chart is isomorphic to \mathbb{R}^n . For instance, the chart given by $y_1 = 1$ has coordinates (x_1, y_2, \dots, y_n) since $x_j = x_1 y_j$ for $j > 1$. We can also describe the blowup map in terms of its restriction to these affine charts. Let $f(x)$ be the defining equation of a hypersurface V in \mathbb{R}^n . Then, in the affine chart $y_1 = 1$, the blowup of V is described by the equation

$$f(x_1, x_1 y_2, x_1 y_3, \dots, x_1 y_n) = 0.$$

This equation is known as the total transform of $f(x)$.

The perspective transform shows a lot of similarities to the blowup map because the perspective transform of $f(x)$ is given by

$$f\left(\frac{1}{x_1}, \frac{y_2}{x_1}, \frac{y_3}{x_1}, \dots, \frac{y_n}{x_1}\right).$$

For this reason, it was hoped that singularities on the boundaries of convex semialgebraic sets could be simplified using the perspective transform. This simplification would then be an important step towards proving the Helton-Nie conjecture that all convex semialgebraic sets are spectrahedral shadows. However, as seen in Section 2, perspective transforms are just linear transforms when applied to cones in \mathbb{S}^n . Thus, when restricted to the affine chart $x_0 = 1$ of \mathbb{S}^n , such a transform moves the singularity to a point at infinity. The singularity is not simplified, so we still cannot employ Theorem 3.4 on the transformed set.

One question remains: are there any rational transformations of \mathbb{S}^n which simplify singularities while preserving the convexity of all convex subsets? The answer is no, because a transform which preserves convexity must be a linear transform, and linear transforms do not simplify singularities. Blowups, by definition, separate lines through the origin, and in so doing, they destroy convexity. Moreover, even if the blowup of a given convex set S is a spectrahedral shadow, there is no guarantee that we can construct a representation of S as a spectrahedral shadow from that of its blowup.

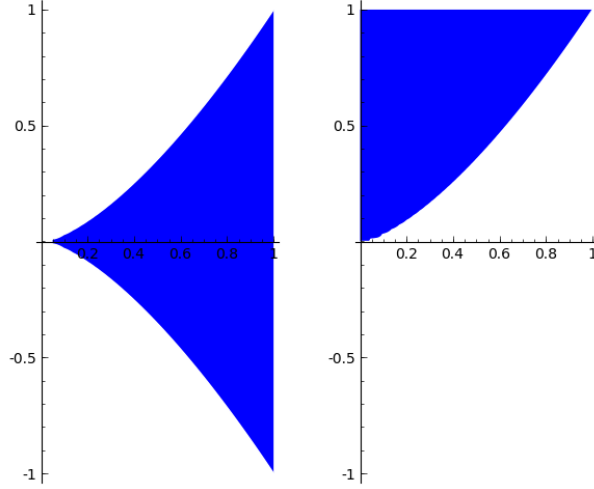
Given these difficulties, we propose a few possible directions for solving the Helton-Nie conjecture.

4.1 Special Structures of Defining Inequalities

In [5, §4], Nie was able to use the perspective transform effectively in representing a convex set with singularity as a spectrahedral shadow. The success of his method can be attributed to [5, Theorem 4.2] which constructs representations of certain convex sets as spectrahedral shadows given special structures in the defining inequalities. As a future extension of this project, it would be interesting to study how this theorem can be generalized using linear transforms of cones in \mathbb{S}^n .

We need also to study the structure of a defining inequality $g(x) \geq 0$ at a singular boundary point u . For instance, in [3, Theorem 3.5], it was shown that if $g(x)$ is irredundant and nonsingular at u , then $g(x)$ is quasi-concave

at u . No such necessary conditions are known for singular points u . To reveal the subtlety of this problem, let us consider the following example. The set in \mathbb{R}^2 defined by the cuspidal cubic $y^2 - x^3 \leq 0$ is not convex at the origin. But the set defined by $\{y^2 - x^3 \geq 0, y \geq 0\}$ is. If we allow nonpolynomial inequalities, then the latter set may be defined by $\{y - x^{3/2} \geq 0, y \geq 0\}$, and $g(x, y) = y - x^{3/2}$ is quasi-convex at the origin.



For smooth points with zero curvature, quasi-convexity can be studied in relation to the Positivstellensatz. Vaguely speaking, given a Positivstellensatz certificate of quasi-convexity, can we convert this certificate into a representation of the convex set as a spectrahedral shadow?

4.2 Duality and Theta bodies

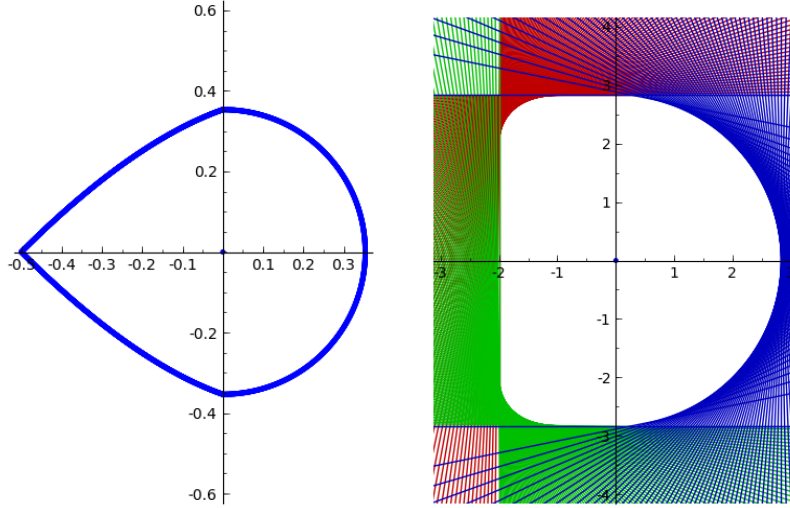
While there are no transforms of \mathbb{S}^n which preserve convexity and simplify singularities, we may still consider duality. Strictly speaking, taking the dual of a convex set does not come from a transform of \mathbb{S}^n . Duality is potentially useful because it preserves convexity and spectrahedral shadows. Also, given Theorem 3.3, we want to apply duality to *local* neighborhoods of singularities on the boundaries of convex sets.

We did not have time to explore this idea, but we shall give an example. Consider the convex set defined by the nodal cubic $x^2 - x^3 - y^2 \geq 0$. We will dualize a neighborhood of the singularity at the origin with respect to

the point $(1/2, 0)$. In the diagrams below, we translated the origin, so that the convex set S we are dualizing is a union of sets

$$T_1 = \{(x, y) \in \mathbb{R}^2 \mid (x + \frac{1}{2})^2 - (x + \frac{1}{2})^3 - y^2 \geq 0, x < 0\}$$

$$T_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq \frac{1}{8}, x > 0\}.$$



The dual body is on the right. We did not compute its defining equations, but one can see that the singular point has been replaced by a smooth point with zero curvature. This might be useful, because as mentioned in Section 4.1, necessary conditions for smooth points with zero curvature are known.

Related to the idea of duality are theta bodies. All theta bodies are spectrahedral shadows. Given a singularity neighborhood of a convex body, one may try to represent the neighborhood as a theta body.

Acknowledgements I would like to thank Bernd Sturmfels for suggesting this topic, and for teaching his insights into the algebraic geometry of convex optimization in the course Math 275. My appreciation also goes to Jiawang Nie, Philipp Rostalski and Raman Sanyal for many useful discussions.

References

- [1] H. Hironaka: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, *Ann. of Math.* (2) **79** (1964) 109–203.
- [2] W. Helton and J. Nie: Semidefinite representation of convex sets, *Mathematical Programming* **122** (2010) 21–64.
- [3] W. Helton and J. Nie: Sufficient and Necessary Conditions for Semidefinite Representability of Convex Hulls and Sets, *SIAM J. on Optimization* **20** (2009) 759–791.
- [4] J. Lasserre: Convex sets with semidefinite representation, *Mathematical Programming* **120** (2009) 457–477.
- [5] J. Nie: First order conditions for semidefinite representations of convex sets defined by rational or singular polynomials. *Mathematical Programming*, to appear.