# Computing Resolutions: How and Why

Shaowei Lin (UC Berkeley)

shaowei@math.berkeley.edu

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Math 255 Algebraic Curves

# Who am I?

- Algebraic statistics

  PhD mathematics (May 2011), advisor: Bernd Sturmfels
  “Algebraic methods for evaluating integrals in Bayesian statistics”
  `http://math.berkeley.edu/~shaowei/`

- Singular learning theory

  Sumio Watanabe, resolution of singularities

- Machine learning

  Collaboration with Stanford artificial intelligence lab (Andrew Ng)
  Unsupervised deep learning

# Why?

# Integral Asymptotics

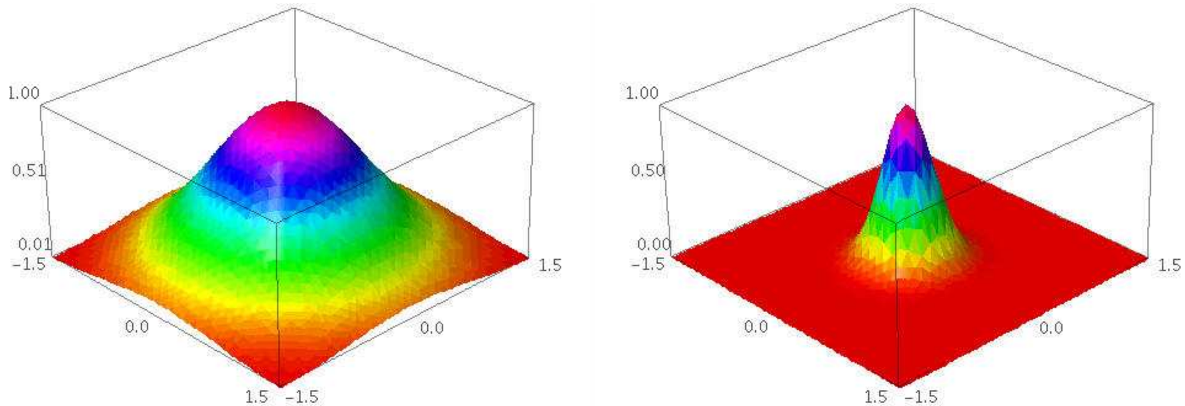
Stirling's approximation.

$$N! \;=\; N^{N+1}\int_0^\infty e^{-N(x-\log x)}\,dx \;\approx\; N^{N+1}\sqrt{\frac{2\pi}{N}}e^{-N}$$

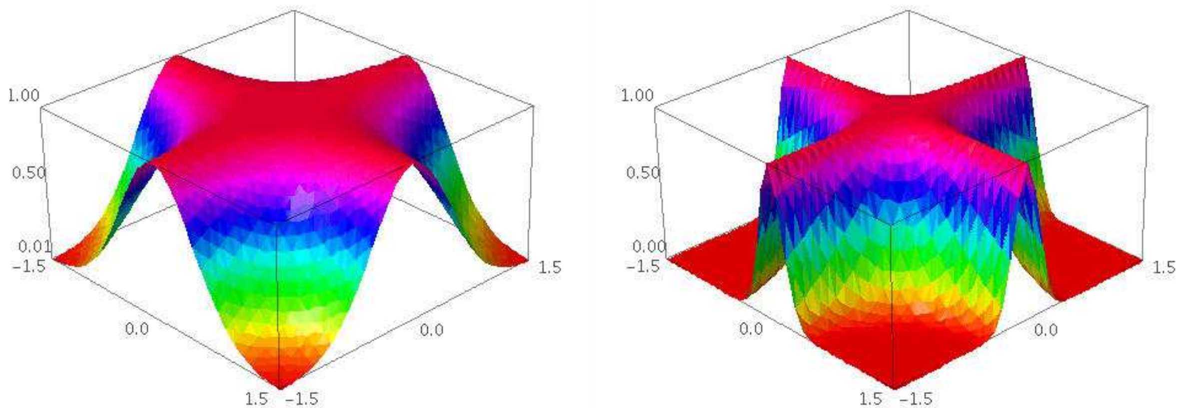
Laplace integrals.

$$Z(N) = \int_\Omega e^{-Nf(\omega)}\varphi(\omega)d\omega$$

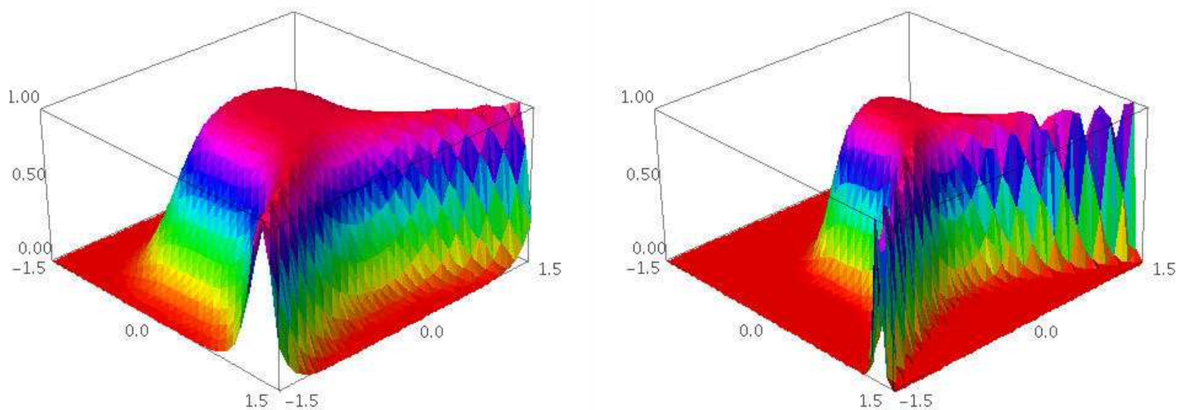
# **Integral Asymptotics** $Z(N) = \int_\Omega e^{-Nf(\omega)}\varphi(\omega)d\omega$




$$f(x,y) = x^2 + y^2$$




$$f(x,y) = (xy)^2$$




$$f(x,y) = (y^2 - x^3)^2$$

# Integral Asymptotics

Statistical integral. $\displaystyle\int_{[0,1]^2} (1 - x^2y^2)^{N/2}\, dxdy \;\approx$

$$\sqrt{\frac{\pi}{8}} N^{-\frac{1}{2}} \log N \qquad -\sqrt{\frac{\pi}{8}}\left(\frac{1}{\log 2} - 2\log 2 - \gamma\right) N^{-\frac{1}{2}}$$

$$-\frac{1}{4} N^{-1} \log N \qquad +\frac{1}{4}\left(\frac{1}{\log 2} + 1 - \gamma\right) N^{-1}$$

$$-\frac{\sqrt{2\pi}}{128} N^{-\frac{3}{2}} \log N \qquad +\frac{\sqrt{2\pi}}{128}\left(\frac{1}{\log 2} - 2\log 2 - \frac{10}{3} - \gamma\right) N^{-\frac{3}{2}}$$

$$0 \qquad -\frac{1}{24} N^{-2} \qquad + \cdots$$

Euler-Mascheroni constant $\displaystyle\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.5772156649.$

# Log Canonical Thresholds

$I = \langle f_1, \ldots, f_r \rangle \subset k[\omega_1, \ldots, \omega_d]$ ideal
$k$ favorite field, $W \subset k^d$ small nbd of origin.

*Complex LCT* $\lambda$: smallest pole of the zeta function

$$\zeta(z) = \int_W \left( |f_1(\omega)|^2 + \cdots + |f_r(\omega)|^2 \right)^{-z} d\omega, \quad z \in \mathbb{C}.$$

*Real LCT* $\lambda$: smallest pole of the zeta function

$$\zeta(z) = \int_W \left( f_1(\omega)^2 + \cdots + f_r(\omega)^2 \right)^{-z/2} d\omega, \quad z \in \mathbb{C}.$$

Independent of choice of generators!

*Real LCT* $\lambda$: coefficient in asymptotics

$$Z(N) = \int_W e^{-N(f_1(\omega)^2 + \cdots + f_r(\omega)^2)} \, d\omega \approx C N^{-\lambda} (\log N)^{\theta - 1}.$$

# How?

# Blowups: Definition

Let $Z \subset \mathbb{A}^d$ be a smooth variety whose ideal is $\langle f_1, \ldots, f_r \rangle$.
e.g. $Z$ is the origin, whose ideal is $\langle x_1, \ldots, x_d \rangle$.

Let us blow up $\mathbb{A}^d$ with *center* $Z$.

For each point $x = (x_1, \ldots, x_d) \in \mathbb{A}^d$ not in $Z$, let us tag it with the point $f^{\mathbb{P}}(x) = (f_1(x) : \cdots : f_r(x)) \in \mathbb{P}^{r-1}$. The set $X$ of points

$$(x, f^{\mathbb{P}}(x)) \in \mathbb{A}^d \times \mathbb{P}^{r-1}, \quad x \in \mathbb{A}^d \backslash Z$$

has a Zariski closure $\widetilde{X}$ called the *blowup* of $\mathbb{A}^d$ with center $Z$.

The projection $\pi : \widetilde{X} \subset \mathbb{A}^d \times \mathbb{P}^{r-1} \to \mathbb{A}^d$ is the *blowup map*.
This map restricts to an isomorphism $X \to \mathbb{A}^d \backslash Z$, while the preimage $E = \pi^{-1} Z \simeq Z \times \mathbb{P}^{\mathrm{codim(Z)}-1}$ is the *exceptional divisor*.

# Blowups: Properties

- $\widetilde{X}$ is smooth and is the union $X \cup E$.
- $\widetilde{X}$ is the variety

$$\left\{(x,y) \in \mathbb{A}^d \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x) \text{ for all } i,j\right\}.$$

If the center $Z$ is the origin, then $\widetilde{X}$ is a *toric* variety.

- $\widetilde{X}$ is covered by affine charts $U_i = \{(x,y) \in X : y_i \neq 0\}$ where

$$U_i \simeq \mathrm{Spec}\ k[x_1, \ldots, x_d, \frac{f_1(x)}{f_i(x)}, \ldots, \frac{f_r(x)}{f_i(x)}].$$

If the center $Z$ is the origin, then $U_i \simeq \mathbb{A}^d$ and it has coordinates

$$(\xi_1, \ldots, \xi_d) = (\frac{y_1}{y_i}, \ldots, x_i, \ldots, \frac{y_d}{y_i}).$$

The blowup map $\pi$ restricts to the affine map $\pi_i : U_i \to \mathbb{A}^d$,

$$(\xi_1, \ldots, \xi_d) \mapsto (\xi_1\xi_i, \ldots, \xi_i, \ldots, \xi_d\xi_i).$$

# Blowups: Transforms

Let $V \subset \mathbb{A}^d$ be a variety which contains a smooth subvariety $Z$.
Let $\pi : \widetilde{X} \to \mathbb{A}^d$ be the blowing up of $\mathbb{A}^d$ with center $Z$.

*Total* transform: $\pi^{-1}(V)$

*Strict* (or proper) transform: $\pi_{\mathrm{st}}^{-1}(V) = \overline{\pi^{-1}(V \setminus Z)}$.

Exceptional divisor: $E = \pi^{-1}(Z)$

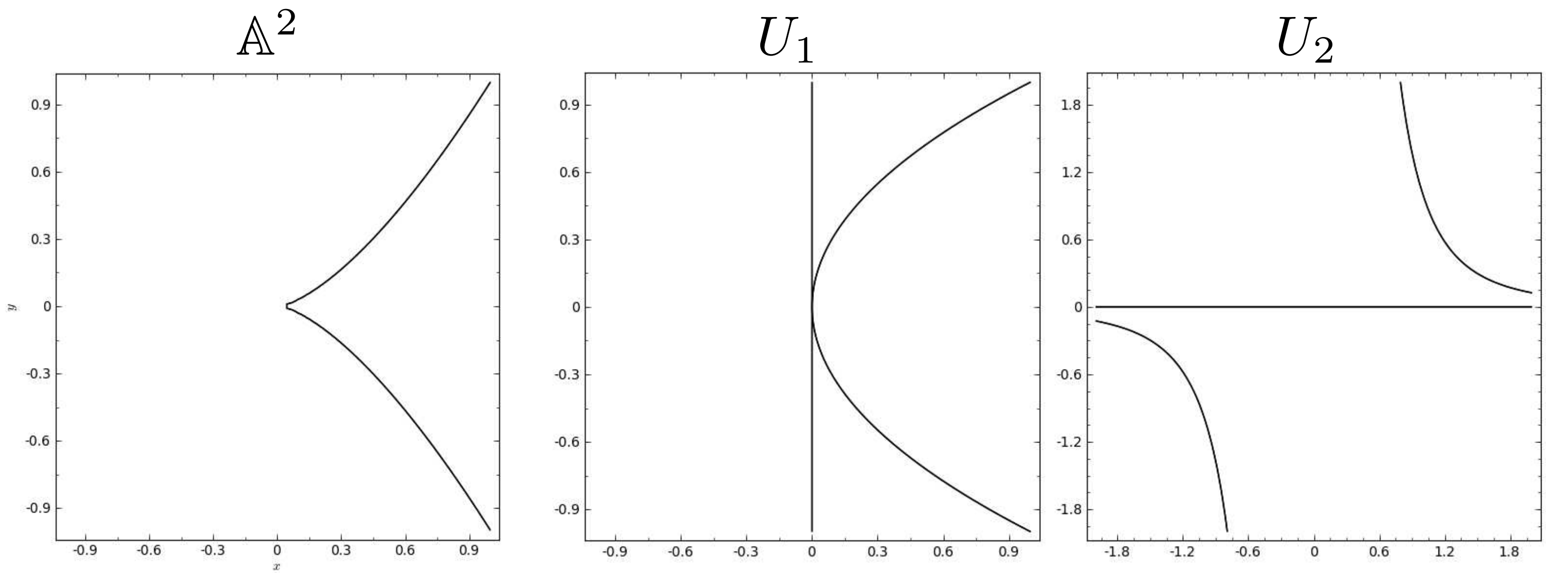
$$\pi^{-1}(V) \ = \ \pi_{\mathrm{st}}^{-1}(V) \ \cup \ E$$

# Blowups: Example

Suppose we blow up the origin in $\mathbb{A}^2$. The blowup $\widetilde{X}$ can be thought of as a *mobius strip* with infinite width. It has two charts $U_1, U_2$.

Consider the cusp $\mathcal{V}(y^2 - x^3) \subset \mathbb{A}^2$. Under the blowing up,

- $\pi_1 : U_1 \to \mathbb{A}^2, (s,t) \mapsto (x,y) = (s, st). \quad y^2 - x^3 = s^2(t^2 - s)$
- $\pi_2 : U_2 \to \mathbb{A}^2, (u,v) \mapsto (x,y) = (uv, v). \quad y^2 - x^3 = v^2(1 - u^3 v)$



# Software

# Singular Code

```
LIB "sing.lib";
LIB "primdec.lib";
LIB "resolve.lib";
LIB "reszeta.lib";
LIB "resgraph.lib";

ring RR = 0, (x,y), dp;
ideal I = y^2-x^3;
ideal Z = x,y;
list L = blowUp(I,Z);
size(L);
def Q = L[1]; setring Q;
showBO(BO);  // BO = Basic Object
setring RR;
```

# Singular Code

```
// resolve: "A" = All charts,
//          "L" = Locally at origin

ideal I = y^2-x^3;
list L = resolve(I, 0,"A","L");
presentTree(L);

ideal I = x^6+y^6-x*y;
list L = resolve(I, 0,"A","L");
presentTree(L);

ideal I = x^6+y^6+y^2-x^3;
list L = resolve(I, 0,"A","L");
presentTree(L);
```

# Resolution of Singularities

**For curves**:

Blow up singularities (points) one by one.

**For varieties of higher dimension**:

Blow up smooth subvarieties of singular locus.
Not easy to identify these smooth centers in general.
Knowing how to do this will get you a Field's Medal.
Dimension reduction via *hypersurfaces of maximal contact*

# Other Cool Facts

- Blowing up a linear subspace

  Let $\pi : \widetilde{X} \to \mathbb{R}^n$ be the blowing up of the origin in $\mathbb{R}^n$. Then, $\pi \times \mathrm{id} : \widetilde{X} \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ is the blowing up of $\{0\} \times \mathbb{R}^m$ in $\mathbb{R}^{n+m}$.

- Global blowups is the gluing of local blowups

  Suppose we want to blow up a smooth center $Z$ in $\mathbb{R}^n$. Cover $\mathbb{R}^n$ with small affine charts, and pick coordinates in each chart so that $Z$ is a linear subspace of each chart. Then, blowing up $Z$ in $\mathbb{R}^n$ is equivalent to blowing up the corresponding linear subspaces in each chart and gluing these maps together.

**Moral of the Story**: We only need to know how to blow up the origin!