Relations among Principal Minors of a Matrix

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- principal minor: a minor with rows and columns indexed by same subset $\sigma \subset [n] := \{1, \ldots, n\}$.
- $A_\sigma \in \mathbb{C}$: principal minor of A indexed by σ.

$$\sigma = 124 \qquad \begin{pmatrix} a_{11} & a_{12} & \cdot & a_{14} \\ a_{21} & a_{22} & \cdot & a_{24} \\ \cdot & \cdot & \cdot & \cdot \\ a_{41} & a_{42} & \cdot & a_{44} \end{pmatrix} \qquad A_{124} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

Definitions

- $A_* \in \mathbb{C}^{2^n-1}$: the vector of principal minors of A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{aligned} A_* &= [A_1, A_2, A_3, A_{12}, A_{13}, A_{23}, A_{123}] \\ &= [1, 5, 9, -3, -12, -3, 0] \end{aligned}$$

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- $P_n \in \mathbb{C}[A_*]$: prime ideal of all polynomial relations among the principal minors, $P_n = \mathcal{I}(\text{Im}\phi)$.

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- **Main Problem**: Find a finite generating set for P_n.

Motivation

1. Plücker relations

- Relations among minors p_σ of matrix
- p_σ: given subset $\sigma \in [n]$
  choose rows $1, \ldots, d$ and columns $\sigma_1, \ldots, \sigma_d$ where $d = |\sigma|$.

$$\sigma = 124 \qquad \begin{pmatrix} a_{11} & a_{12} & \cdot & a_{14} \\ a_{21} & a_{22} & \cdot & a_{24} \\ a_{31} & a_{32} & \cdot & a_{34} \\ . & . & . & . \end{pmatrix} \qquad p_{124} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}$$

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- Also has a total of $2^n - 1$ minors.
- Prime ideal of relations is generated by quadrics.

Motivation

2. Matrix Theory & Probability Theory

   Information about principal minors important in

   - Detection of P-matrices, GKK-matrices
   - Inverse multiplicative eigenvalue problem
   - Determinantal point processes
   - Negatively-correlated random variables

Motivation

3. Principal Minor Assignment Problem [PMAP]

- In many applications, desirable to know when a vector of length $2^n - 1$ is realizable as the principal minors of some $n \times n$ matrix.
- Formulated as open problem by Holtz & Schneider [4], 2001.
- Solved *algorithmically* for "generic" case by Kent & Tsatsomeros [5], 2006.

Motivation

- Principal minors of Symmetric Matrix
  - Holtz & Sturmfels [3], 2007: studied relations among principal minors of a *symmetric* matrix.
  - Found generating set for 4×4 case consisting of 20 quartics.
  - Conjectured generating set for general $n \times n$ case.
  - If conjecture is true, it *algebraically* solves PMAP for symmetric matrices.

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- Algorithmically:
  - Either produces a matrix realizing the vector of principal minors, or determines that none exist.
- Algebraically:
  - PMAP $\cong$ "Is A_* in Imϕ?"
  - By Chevalley's Theorem, Imϕ is constructible, i.e.

$$\text{Im}\phi = X_0 - X_1 + \cdots + (-1)^k X_k$$

    where $X_0 \supset X_1 \supset \cdots \supset X_k$ are varieties.
  - PMAP solved if we find finite generating sets for $X_1, \ldots, X_k$.
  - Matrix realizing principal minors is NOT produced.

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  then A_* is realized by

$$A = \begin{pmatrix} x & xy - z \\ 1 & y \end{pmatrix}$$

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- $n = 3$: (Exercise) Show that all vectors A_* are realizable.

Thus, $P_1 = P_2 = P_3 = \{0\}$.

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Find a finite generating set for P_n.

- $n = 4$:
  - (Exercise) This vector of principal minors is not realizable:
    $A_{123} = A_{124} = A_{134} = A_{234} = 1, A_1 = A_2 = \ldots = A_{1234} = 0$
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    Thus, $P_4 \neq \{0\}$.
  - Using Gröbner basis methods via `Singular`.
  - E.J. Nanson & Thomas Muir, 1897:
    found relations in P_4 using **devertebrated minors**.

Devertebrated Minors

Given a matrix A, replace diagonal elements by zeroes.

Call this new matrix B.

$$B = \begin{pmatrix} \cdot & a_{12} & a_{13} & a_{14} \\ a_{21} & \cdot & a_{23} & a_{24} \\ a_{31} & a_{32} & \cdot & a_{34} \\ a_{41} & a_{42} & a_{43} & \cdot \end{pmatrix}$$

The principal minors $B_\sigma, |\sigma| > 1$ are the **devertebrated minors** of A.

$$B_{23} = \begin{pmatrix} \cdot & a_{23} \\ a_{32} & \cdot \end{pmatrix} = -a_{23}a_{32} = A_{23} - A_2 A_3.$$

Devertebrated Minors

Prop.(Cayley) The principal minors of A are polynomial function of the devertebrated minors and diagonal elements.

$$A_\sigma = \sum_{S \sqcup T = \sigma} B_S \prod_{t \in T} A_t$$

Conversely,

Prop.(Muir) The devertebrated minors (and diagonal elements) of A are polyomial functions of the principal minors.

$$B_\sigma = \sum_{S \sqcup T = \sigma} (-1)^{|T|} A_S \prod_{t \in T} A_t$$

$\therefore$ relations btwn devertebrated minors $\Rightarrow$ relations btwn principal minors

Trigo Trick

Lemma: If $x/y = e^{2i\theta}$ where $x, y, \theta, \in \mathbb{C}$, then $x + y = 2\sqrt{xy}\cos\theta$.

Notation: $[i_1 i_2 \ldots i_k] := a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}$.

Define $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that

$$\frac{[234]}{[243]} = e^{2i\alpha}$$
$$\frac{[134]}{[143]} = e^{2i\beta}$$
$$\frac{[124]}{[142]} = e^{2i\gamma}$$
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Define $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that

$$\frac{[234]}{[243]} = e^{2i\alpha} \quad \Rightarrow \quad B_{234} = [234] + [243] = 2\sqrt{-B_{23}B_{34}B_{42}}\cos\alpha$$
$$\frac{[134]}{[143]} = e^{2i\beta} \quad \Rightarrow \quad B_{134} = [134] + [143] = 2\sqrt{-B_{13}B_{34}B_{41}}\cos\beta$$
$$\frac{[124]}{[142]} = e^{2i\gamma} \quad \Rightarrow \quad B_{124} = [124] + [142] = 2\sqrt{-B_{12}B_{24}B_{41}}\cos\gamma$$
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\end{aligned}$$

Note that $\delta = \alpha + \beta + \gamma$. Similarly,

$$\begin{aligned}
B_{1234} = B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23} - 2\sqrt{B_{12}B_{24}B_{43}B_{31}}\cos(\beta+\gamma) \\
-2\sqrt{B_{12}B_{23}B_{34}B_{41}}\cos(\alpha+\gamma) - 2\sqrt{B_{14}B_{42}B_{23}B_{31}}\cos(\alpha+\beta)
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\end{aligned}$$

Idea: Eliminate α, β, γ from the above 5 equations.

Trigo Trick

Now, using the expansion

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos(\beta+\gamma) + \cos\beta\cos(\alpha+\gamma) + \cos\gamma\cos(\alpha+\beta) - 2\cos\alpha\cos\beta\cos\gamma,$$

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we can rewrite the 5 equations in matrix form:

$$\begin{pmatrix} B_{123}B_{14} & B_{124}B_{13} & B_{134}B_{12} & 2B_{234}B_{12}B_{13}B_{14} - B_{134}B_{124}B_{123} \\ B_{124}B_{23} & B_{123}B_{24} & B_{234}B_{21} & 2B_{134}B_{21}B_{23}B_{24} - B_{234}B_{124}B_{123} \\ B_{134}B_{32} & B_{234}B_{31} & B_{123}B_{34} & 2B_{124}B_{31}B_{32}B_{34} - B_{234}B_{134}B_{123} \\ B_{234}B_{41} & B_{134}B_{42} & B_{124}B_{43} & 2B_{123}B_{41}B_{42}B_{43} - B_{234}B_{134}B_{124} \\ 1 & 1 & 1 & B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23} - B_{1234} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

$$\begin{aligned} x &= -\sqrt{B_{12}B_{24}B_{43}B_{31}}\cos(\beta+\gamma) \\ y &= -\sqrt{B_{12}B_{23}B_{34}B_{41}}\cos(\alpha+\gamma) \\ z &= -\sqrt{B_{14}B_{42}B_{23}B_{31}}\cos(\alpha+\beta) \end{aligned}$$

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Now, using the expansion

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos(\beta+\gamma)+\cos\beta\cos(\alpha+\gamma)+\cos\gamma\cos(\alpha+\beta)-2\cos\alpha\cos\beta\cos\gamma,$$

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Each 4×4 sub-matrix has a non-trivial null space.

Thus, the determinant of each 4×4 sub-matrix vanishes.

Conclusion

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Each maximal minor of the above matrix gives a relation in P_4.

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Recently, we showed Imϕ is closed for all $n > 0$, and that these relations do not generate all of P_4 (see counter-example).

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Open Questions:

- What additional relations are there in P_4?
- Can we extend the above methods (devertebrated minors & trigonometry) to find relations in $P_n, n > 4$.

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